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On the Theory of Consistence of Logical Class-Frequencies, and Its Geometrical Representation

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III. *On the Theory of Consistence of Logical Class-frequencies, and its Geometrical Representation.*

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Contents.

§ 1. In the ordinary treatment of logic the field of discussion is strictly limited to premises of a non-numerical character, numerically definite data being rigidly excluded. The statistician obtains no help from ordinary logic towards solving even the simplest problems, *e.g.*, the deduction of inferences from data of the type "x per cent. of A's are B, y per cent. of A's are C," or the inferring of association between B and C from known associations of A with B and with C.

It is now more than half a century since De Morgan, in the chapter "On the Numerically Definite Syllogism," of his 'Formal Logic' (1847), laid the foundations of a theory of strictly quantitative logic. Substituting the modified notation of Jevons, employed by me in a recent paper,* for De Morgan's own notation, his Theorem may be expressed in the form "if (AB) + (AC) > (A), (BC) must be *at least* equal to the difference (AB) + (AC) − (A)." For if, *e.g.*, we imagine (A) boxes, into which

* "On the Association of Attributes in Statistics," &c., 'Phil. Trans.,' A, 1900, vol. 194, p. 257. The notation is essentially that of Jevons, save that small Greek letters have been substituted for his italics.

we deal a certain number (viz., (AB)) of cards marked (B), one into each box, leaving only (A) − (AB) unoccupied boxes, and then proceed to deal into the remaining boxes cards marked C, one into each; some of these must fall into boxes already occupied by a B if their number exceed (A) − (AB). From similar simple reasoning De Morgan derived the complete conditions of consistence for (AB), (AC), and (BC).* He does not, however, consider the case of more than three attributes, and the whole discussion is rendered very lengthy owing to his standpoint being still that of the older logicians.

The theory of numerical logic was carried somewhat further by Boole in Chapter 19 of the 'Laws of Thought' (1854), entitled "Of Statistical Conditions." After taking a series of propositions for finding the major and minor limits to class-frequencies or sums of frequencies of any order, in terms of the first-order frequencies and the total frequency only, he proceeds to the general problem "given the respective numbers of individuals comprised in any classes, s, t, &c., logically defined, to deduce a system of numerical limits of any other class w also logically defined." I must confess myself unable to follow the physical meaning of the processes symbolically developed in Boole's general theorem, and this chapter has not, to my knowledge, been discussed by subsequent logicians. One naturally turns to the "Symbolic Logic" of Dr. Venn, whose lucid treatment clears up many difficulties of the "Laws of Thought," but he does not appear to deal with the problems "of statistical conditions."

In the following memoir I have endeavoured to deal with the general theory of logical consistence, as I prefer to term it, from a standpoint slightly different to that of Boole.†

§ 2. Let (U) be the total frequency in some defined universe, and let

$$(A)\ (B)\ (C)\ .\ .\ .\ (AB)\ (AC)\ .\ .\ .\ (ABC)\ .\ .\ .\ \&c.,$$

be the frequencies of the positive groups (classes) up to, say, groups of the nth order, the number of attributes specified being m ($m > n$). It will be remembered that all other group-frequencies can be expressed in terms of those of the positive groups, so that no others need be considered. Boole, in his general theorem, quoted above, supposes certain of these frequencies to be known, and requires to find the resulting limits to some one other. I propose instead to make, at the outset, no supposition as to the frequencies that are known, but simply to discuss what conditions must hold if the whole set of frequencies is to be self-consistent. By proceeding in this way symmetrical systems of conditions are obtained of great interest and generality. They may be applied at once to such cases of limit-inference as are dealt with by

* See § 5 below.

† I may perhaps state that this work was not directly suggested by De Morgan's or by Boole's writings. Difficulties had arisen in the invention of numerical examples to illustrate certain points of theory, and I was driven to working out the theory of consistence in order to clear up these difficulties.

BOOLE, while exhibiting in a clearer light the data that are necessary for any inference, and the limitations of inference caused by assigned limitations of data.

The whole of the conditions of consistence are derived from one source, or one condition only, viz., that *no frequency can be less than zero.* But as it is evident that if *all* the frequencies of any order be greater than zero, the frequencies of all lower orders must, *à fortiori*, be greater than zero, we may limit the above statement by saying that all the conditions of consistence are covered by the dictum that *no ultimate frequency* can be less than zero.* If, however, we are dealing with groups of the nth and lower orders only in m specified attributes, it is convenient to divide the conditions into two classes—(1) the "inferior conditions of consistence," which may be derived from the fact that no nth order frequency can be less than zero; (2) the "superior conditions of consistence" which can only be derived from the consideration that frequencies of order greater than n cannot be less than zero.

A distinction of this sort is, it may be noted, made by DE MORGAN. Inferences drawn from the *inferior* conditions of consistence for second order groups he terms *spurious* inferences;† they do not really follow from given premises (*i.e.*, given values of (AB) and (AC) or (AB) and (BC), &c.), but are "true by the constitution of the universe."‡

§ 3. It will be convenient to use the following terms in addition to those defined in my previous memoir.

A set of frequencies formed by taking the frequency of any positive group ABCD . . . N, together with the frequencies of all possible groups of the same order formed by substituting the contraries $\alpha\, \beta\, \gamma\, \delta \ldots \nu$ for one or more of the attributes ABCD . . . N, will be called an "aggregate" of frequencies. Any one aggregate contains only one positive group which may be used to denote the aggregate, so that one may speak of the AB aggregate or the ABCD aggregate. The *order* of an aggregate may be defined as the order of the groups contained in it. An aggregate of order n contains 2^n groups. The sum of the frequencies of these 2^n groups of the aggregate, is evidently equal to the total frequency or number of observations (U).

If m attributes be specified, the number of positive groups of the nth order that can be formed from them is

$$\frac{m(m-1)(m-2)\ldots(m-n+1)}{(n)!}$$

The complete set of *consistent* aggregates corresponding to these positive groups will be termed a congruence of aggregates or simply a congruence. The number

* The frequency of a group specified by all the m attributes noted. "On the Association of Attributes," &c., *loc. cit.*, p. 259.

† 'Formal Logic,' p. 153.

‡ Note on same page.

of attributes specified, m, will be termed the *degree* of the congruence, its *order* being the order of the component aggregates.*

A congruence of degree m and order n may be regarded as built up of a series of congruences of degree lower than m. Thus the congruence of the fourth degree and second order containing the aggregates AB, AC, AD, BC, BD, CD, may be split up into four congruences of the third degree only, viz., those containing the aggregates (AB, AC, BC) (AB, AD, BD) (AC, AD, CD) and (BC, BD, CD) respectively. In general, any congruence of the mth degree may be divided into

$$\frac{m(m-1)\ldots(m-r+1)}{r!}$$

congruences of the rth degree, one to each positive group of the rth order that can be formed from the m attributes. These component congruences evidently, as in the above example, overlap: any one aggregate occurs in two or more congruences.

In the following sections (§ 4–§ 10) I proceed to the discussion of congruences of the third, fourth, and fifth degrees; §§ 11–13 deal with the general case, and the remainder of the paper consists of a discussion of the geometrical representation, by means of polyhedra, of the conditions of consistence, together with a few numerical illustrations.

Congruence of the Third Degree.

§ 4. In terms of the definition § 2, p. 93, the inferior conditions of consistence are given at once by expanding all second-order frequencies in terms of the positive groups only, and putting the resulting expression $\not< 0$. Thus, retaining for convenience the second-order terms only on the left of the inequality, we must have

$$\left.\begin{array}{lll} (\mathrm{AB}) \not< 0 & \text{or} & (\mathrm{AB}) \text{ will be negative} \\ \quad\ \not< (\mathrm{A}) + (\mathrm{B}) - (\mathrm{U}) & \text{or} & (\alpha\beta) \quad ,, \quad ,, \\ \quad\ \not> (\mathrm{A}) & \text{or} & (\mathrm{A}\beta) \quad ,, \quad ,, \\ \quad\ \not> (\mathrm{B}) & \text{or} & (\alpha\mathrm{B}) \quad ,, \quad ,, \end{array}\right\} \quad . \quad . \quad . \quad . \quad \mathrm{I.}$$

Similar conditions hold of course for (AC) and (BC).

§ 5. To find the superior conditions of consistence (*cf.* again the definition), write down the *inferior* conditions for the (ABC) aggregate. These are of course given similarly by expanding the third-order frequencies in terms of the positive groups, and putting the resulting expansion $\not< 0$. Thus—

* In any systematic tabulation of frequencies I think the grouping should be made by aggregates. Such an arrangement would be distinctly better than that adopted by me in the sample table on p. 318 of my paper on "Association," *loc. cit.*

	Or the frequency given below will be negative.		
(ABC) $\not<$ 0	(ABC)	[1]	II.
$\not<$ (AB) + (AC) − (A)	(A$\beta\gamma$)	[2]	
$\not<$ (AB) + (BC) − (B)	(αBγ)	[3]	
$\not<$ (AC) + (BC) − (C)	($\alpha\beta$C)	[4]	
$\not>$ (AB)	(ABγ)	[5]	
$\not>$ (AC)	(AβC)	[6]	
$\not>$ (BC)	(αBC)	[7]	
$\not>$ (AB) + (AC) + (BC) − (A) − (B) − (C) + (U)	($\alpha\beta\gamma$)	[8]	

Now evidently, any third-order aggregate whatever is impossible *if any one of the minor limits* [1]–[4] *be greater than any one of the major limits* [5]–[8]. If the second-order frequencies be such as to create this condition they must be impossible within one and the same universe, *i.e.*, they are inconsistent or incongruent. There are sixteen comparisons to be made, taking each of the major limits in turn with each of the minor limits, but the majority of these comparisons, viz., 12, only lead back to the inferior conditions. The four comparisons of expansions due to contrary frequencies alone lead to new conditions—

II. [1] [8]	(AB) + (AC) + (BC) $\not<$ (A) + (B) + (C) − (U)	[1]	III.
II. [2] [7]	− (AB) + (AC) + (BC) $\not>$ (C)	[2]	
II. [3] [6]	(AB) − (AC) + (BC) $\not>$ (B)	[3]	
II. [4] [5]	(AB) + (AC) − (BC) $\not>$ (A)	[4]	

These are the four superior conditions of consistence for the congruence of the third degree. In order that the second-order frequencies may be consistent with each other and with the given frequencies of the first order, they must fulfil all the conditions of type I. and the conditions of type III. These inequalities are highly interesting; but discussion is best deferred till after we have obtained the similar conditions for the congruences of higher degree.

Congruence of the Fourth Degree.

§ 6. Two congruences of this degree are possible, viz., those of the second and third orders. The third-order congruence will be taken first, as the conditions of consistence for the second-order congruence may be obtained directly from the third-order conditions.

The inferior conditions need not be written down as they have been given already (II. of § 5); similar conditions hold of course for all four groups (ABC), (ABD), (ACD), (BCD).

§ 7. To obtain the superior conditions, the fourth-order groups must be expanded (*cf.* § 5), and the expansions again put $\nless 0$, thus giving the systems of limits for (ABCD) as below.

Limits for (ABCD)	Or the frequency given below will be negative.	
$(ABCD) \nless 0$	$(ABCD)$	[1]
$\nless (ACD) + (BCD) - (CD)$	$(\alpha\beta CD)$	[2]
$\nless (ABD) + (BCD) - (BD)$	$(\alpha B\gamma D)$	[3]
$\nless (ABC) + (BCD) - (BC)$	$(\alpha BC\delta)$	[4]
$\nless (ABD) + (ACD) - (AD)$	$(A\beta\gamma D)$	[5]
$\nless (ABC) + (ACD) - (AC)$	$(A\beta C\delta)$	[6]
$\nless (ABC) + (ABD) - (AB)$	$(AB\gamma\delta)$	[7]
$\nless (ABC) + (ABD) + (ACD) + (BCD) - (AB) - (AC) - (AD) - (BC) - (BD) - (CD) + (A) + (B) + (C) + (D) - (U)$	$(\alpha\beta\gamma\delta)$	[8]
$\ngtr (ABC)$	$(ABC\delta)$	[9]
$\ngtr (ABD)$	$(AB\gamma D)$	[10]
$\ngtr (ACD)$	$(A\beta CD)$	[11]
$\ngtr (BCD)$	(αBCD)	[12]
$\ngtr (ABD) + (ACD) + (BCD) - (AD) - (BD) - (CD) + (D)$	$(\alpha\beta\gamma D)$	[13]
$\ngtr (ABC) + (ACD) + (BCD) - (AC) - (BC) - (CD) + (C)$	$(\alpha\beta C\delta)$	[14]
$\ngtr (ABC) + (ABD) + (BCD) - (AB) - (BC) - (BD) + (B)$	$(\alpha B\gamma\delta)$	[15]
$\ngtr (ABC) + (ABD) + (ACD) - (AB) - (AC) - (AD) + (A)$	$(A\beta\gamma\delta)$	[16]

IV.

Any fourth-order aggregate will be impossible if any one of the minor limits to (ABCD) be greater than any one of the major limits. There are eight of each, or sixty-four comparisons to be made; but thirty-two of these lead only to the already known inferior conditions of consistence. The remaining thirty-two, involving comparisons of pairs of groups that are contrary as regards *three* attributes (as in § 5), give new conditions all obtainable by cyclic substitution from the eight given in V. below.

There are four third-order positive groups to be formed from four attributes, and therefore four possible sets of three, giving four sets of the eight inequalities V. The remaining three sets may be at once written down by substituting A, B, or C for D (and conversely) in the set given below.

$$\left.\begin{array}{lll}
(ABD) + (ACD) + (BCD) & \not> (AB) + (AC) + (AD) + (BC) + (BD) + (CD) - (A) - (B) - (C) - (D) + (U) & [1] \\
& \not< (AD) + (BD) + (CD) - (D) & [2] \\
(ABD) + (ACD) - (BCD) & \not> (AD) & [3] \\
& \not< (AB) + (AC) + (AD) - (BC) - (A) & [4] \\
(ABD) - (ACD) + (BCD) & \not> (BD) & [5] \\
& \not< (AB) - (AC) + (BC) + (BD) - (B) & [6] \\
-(ABD) + (ACD) + (BCD) & \not> (CD) & [7] \\
& \not< -(AB) + (AC) + (BC) + (CD) - (C) & [8]
\end{array}\right\} \text{V.}$$

§ 8. The superior conditions of consistence for the *second*-order congruence of the fourth degree may be at once obtained from V. For in order that the congruence of the third order may be self-consistent, it is clearly necessary that the congruence of the second order should be so; if this condition do not hold the conditions of consistence V. will prove themselves impossible. But the limits V. only become impossible if, in either of the four pairs of limits ([1] [2], [3] [4] &c.), the minor limit be greater than the major. If we express the condition that each minor limit must be the less we have,

$$\begin{array}{ll}
\text{from V [1] [2]} & (AB) + (AC) + (BC) \not> (A) + (B) + (C) - (U). \\
\text{[3] [4]} & (AB) + (AC) - (BC) \not< (A). \\
\text{[5] [6]} & (AB) - (AC) + (BC) \not< (B). \\
\text{[7] [8]} & -(AB) + (AC) + (BC) \not< (C).
\end{array}$$

But these are simply the conditions III. of § 5, the superior conditions of consistence for the second-order congruence of the *third* degree. Similar conditions for the aggregates ABD, ACD, BCD, will of course be derived from the unwritten sets of inequalities corresponding to (V.). The theorem therefore holds—"A congruence of the fourth degree and second order is self-consistent, if each of the three congruences of the third degree, into which it can be resolved, is self-consistent."

Congruence of the Fifth Degree.

§ 9. Congruences of the fifth degree may be of either the second, third, or fourth order. The congruence of the fourth order may be taken first and the others derived from it.

The inferior conditions of consistence for fourth-order groups have been already given in § 7, IV. The similar conditions for fifth-order groups, obtained of course in precisely the same way, are as follows:—

		Or the frequency given below is negative.	
(ABCDE)	$\nless$ 0	(ABCDE)	[1]
,,	$\nless$ (ACDE) + (BCDE) − (CDE)	$(\alpha\beta CDE)$	[2]
,,	$\nless$ (ABDE) + (BCDE) − (BDE)	$(\alpha B\gamma DE)$	[3]
,,	$\nless$ (ABCE) + (BCDE) − (BCE)	$(\alpha BC\delta E)$	[4]
,,	$\nless$ (ABCD) + (BCDE) − (BCD)	$(\alpha BCD\epsilon)$	[5]
,,	$\nless$ (ABDE) + (ACDE) − (ADE)	$(A\beta\gamma DE)$	[6]
,,	$\nless$ (ABCE) + (ACDE) − (ACE)	$(A\beta C\delta E)$	[7]
,,	$\nless$ ABCD) + (ACDE) − (ACD)	$(A\beta CD\epsilon)$	[8]
,,	$\nless$ (ABCE) + (ABDE) − (ABE)	$(AB\gamma\delta E)$	[9]
,,	$\nless$ (ABCD) + (ABDE) − (ABD)	$(AB\gamma D\epsilon)$	[10]
,,	$\nless$ (ABCD) + (ABCE) − (ABC)	$(ABC\delta\epsilon)$	[11]
,,	$\nless$ (ABCE) + (ABDE) + (ACDE) + (BCDE) − (ABE) − (ACE) − (ADE) − (BCE) − (BDE) − (CDE) + (AE) + (BE) + (CE) + (DE) − (E)	$(\alpha\beta\gamma\delta E)$	[12]
,,	$\nless$ (ABCD) + (ABDE) + (ACDE) + (BCDE) − (ABD) − (ACD) − (ADE) − (BCD) − (BDE) − (CDE) + (AD) + (BD) + (CD) + (DE) − (D)	$(\alpha\beta\gamma D\epsilon)$	[13]
,,	$\nless$ (ABCD) + ABCE) + (ACDE) + (BCDE) − (ABC) − (ACD) − (ACE) − (BCD) − (BCE) − (CDE) + (AC) + (BC) + (CD) + (CE) − (C).	$(\alpha\beta C\delta\epsilon)$	[14]
,,	$\nless$ (ABCD) + (ABCE) + (ABDE) + (BCDE) − (ABC) − (ABD) − (ABE) − (BCD) − (BCE) − (BDE) + (AB) + (BC) + (BD) + (BE) − (B)	$(\alpha B\gamma\delta\epsilon)$	[15]
,,	$\nless$ (ABCD) + (ABCE) + (ABDE) + (ACDE) − (ABC) − (ABD) − (ABE) − (ACD) − (ACE) − (ADE) + (AB) + (AC) + (AD) + (AE) − (A)	$(A\beta\gamma\delta\epsilon)$	[16]
,,	$\ngtr$ (BCDE)	$(\alpha BCDE)$	[17]
,,	$\ngtr$ (ACDE)	$(A\beta CDE)$	[18]
,,	$\ngtr$ (ABDE)	$(AB\gamma DE)$	[19]
,,	$\ngtr$ (ABCE)	$(ABC\delta E)$	[20]
,,	$\ngtr$ (ABCD)	$(ABCD\epsilon)$	[21]
,,	$\ngtr$ (ABDE) + (ACDE) + (BCDE) − (ADE) − (BDE) − (CDE) + (DE)	$(\alpha\beta\gamma DE)$	[22]

VI.

		Or the frequency given below is negative.		
(ABCDE)	$\ngtr$ (ABCE) + (ACDE) + (BCDE) − (ACE) − (BCE) − (CDE) + (CE) . . .	$(\alpha\beta C\delta E)$	[23]	
,,	$\ngtr$ (ABCD) + (ACDE) + (BCDE) − (ACD) − (BCD) − (CDE) + (CD).	$(\alpha\beta CD\epsilon)$	[24]	
,,	$\ngtr$ (ABCE) + (ABDE) + (BCDE) − (ABE) − (BCE) − (BDE) + (BE)	$(\alpha B\gamma\delta E)$	[25]	
,,	$\ngtr$ (ABCD) + (ABDE) + (BCDE) − (ABD) − (BCD) − (BDE) + (BD)	$(\alpha B\gamma D\epsilon)$	[26]	
,,	$\ngtr$ (ABCD) + (ABCE) + (BCDE) − (ABC) − (BCD) − (BCE) + (BC)	$(\alpha BC\delta\epsilon)$	[27]	
,,	$\ngtr$ (ABCE) + (ABDE) + (ACDE) − (ABE) − (ACE) − (ADE) + (AE)	$(A\beta\gamma\delta E)$	[28]	VI.
,,	$\ngtr$ (ABCD) + (ABDE) + (ACDE) − (ABD) − (ACD) − (ADE) + (AD)	$(A\beta\gamma D\epsilon)$	[29]	
,,	$\ngtr$ (ABCD) + (ABCE) + (ACDE) − (ABC) − (ACD) − (ACE) + (AC)	$(A\beta C\delta\epsilon)$	[30]	
,,	$\ngtr$ (ABCD) + (ABCE) + (ABDE) − (ABC) − (ABD) − (ABE) + (AB)	$(AB\gamma\delta\epsilon)$	[31]	
,,	$\ngtr$ (ABCD) + (ABCE) + (ABDE) + (ACDE) + (BCDE) − (ABC) − (ABD) − (ABE) − (ACD) − (ACE) − (ADE) − (BCD) − (BCE) − (BDE) − (CDE) + (AB) + (AC) + (AD) + (AE) + (BC) + (BD) + (BE) + (CD) + (CE) + (DE) − (A) − (B) − (C) − (D) − (E) + (U) . .	$(\alpha\beta\gamma\delta\epsilon)$	[32]	

Any fifth-order aggregate will be impossible if any one of the minor limits to (ABCDE) be greater than any one of the major limits. Since there are sixteen of each, there are 256 comparisons to be made. Eighty of these lead only to the already known inferior conditions. Of the remaining 176, 160 may be obtained by cyclic substitution from the sixteen conditions involving only three fourth-order groups each, as given in VII. below, the remaining sixteen, each involving all five fourth-order positive groups, being given in VIII.

VII.

$$(ABDE) + (ACDE) + BCDE) \ngtr (ABE) + (ACE) + (ADE) + (BCE) + (BDE) + (CDE) - (AE) - (BE) - (CE) - (DE) + (E) \quad [1]$$

$$\ngtr (ABD) + (ACD) + (ADE) + (BCD) + (BDE) + (CDE) - (AD) - (BD) - (CD) - (DE) + (D) \quad [2]$$

$$\nless (ADE) + (BDE) + (CDE) - (DE) \quad [3]$$

$$\nless (ABD) + (ABE) + (ACD) + (ACE) + (ADE) + (BCD) + (BCE) + (BDE) + (CDE) - (AB) - (AC) - (AD) - (AE) - (BC) - (BD) - (BE) - (CD) - (CE) - (DE) + (A) + (B) + (C) + (D) + (E) - (U) \quad [4]$$

$$(ABDE) + (ACDE) - (BCDE) \ngtr (ADE) \quad [5]$$

$$\ngtr (ABD) + (ABE) + (ACD) + (ACE) + (ADE) - (BCD) - (BCE) - (AB) - (AC) - (AD) - (AE) + (BC) + (A) \quad [6]$$

$$\nless (ABE) + (ACE) + (ADE) - (BCE) - (AE). \quad [7]$$

$$\nless (ABD) + (ACD) + (ADE) - (BCD) - (AD) \quad [8]$$

$$(ABDE) - (ACDE) + (BCDE) \ngtr (BDE) \quad [9]$$

$$\ngtr (ABD) + (ABE) - (ACD) - (ACE) + (BCD) + (BCE) + (BDE) - (AB) + (AC) - (BC) - (BD) - (BE) + (B) \quad [10]$$

$$\nless (ABE) - (ACE) + (BCE) + (BDE) - (BE). \quad [11]$$

$$\nless (ABD) - (ACD) + (BCD) + (BDE) - (BD). \quad [12]$$

$$-(ABDE) + (ACDE) + (BCDE) \ngtr (CDE) \quad [13]$$

$$\ngtr -(ABD) - (ABE) + (ACD) + (ACE) + (BCD) + (BCE) + (CDE) + (AB) - (AC) - (BC) - (CD) - (CE) + (C) \quad [14]$$

$$\nless -(ABE) + (ACE) + (BCE) + (CDE) - (CE). \quad [15]$$

$$\nless -(ABD) + (ACD) + (BCD) + (CDE) - (CD). \quad [16]$$

VIII.

$$\left.\begin{aligned}(ABCD) + (ABCE) + (ABDE) + (ACDE) + (BCDE) \not< (ABC) + (ABD) + (ABE) + (ACD) + (ACE)\\ + (ADE) + (BCD) + (BCE) + (BDE) + (CDE) - (AB) - (AC) - (AD) - (AE) - (BC) - (BD)\\ - (BE) - (CD) - (CE) - (DE) + (A) + (B) + (C) + (D) + (E) - (U)\end{aligned}\right\} \quad [1]$$

$$\left.\begin{aligned}(ABCD) + (ABCE) + (ABDE) + (ACDE) - (BCDE) \not> (ABC) + (ABD) + (ABE) + (ACD) + (ACE)\\ + (ADE) - (AB) - (AC) - (AD) - (AE) + (A)\end{aligned}\right\} \quad [2]$$

$$\left.\begin{aligned}(ABCD) + (ABCE) + (ABDE) - (ACDE) + (BCDE) \not> (ABC) + (ABD) + (ABE) + (BCD) + (BCE)\\ + (BDE) - (AB) - (BC) - (BD) - (BE) + (B)\end{aligned}\right\} \quad [3]$$

$$\left.\begin{aligned}(ABCD) + (ABCE) - (ABDE) + (ACDE) + (BCDE) \not> (ABC) + (ACD) + (ACE) + (BCD) + (BCE)\\ + (CDE) - (AC) - (BC) - (CD) - (CE) + (C)\end{aligned}\right\} \quad [4]$$

$$\left.\begin{aligned}(ABCD) - (ABCE) + (ABDE) + (ACDE) + (BCDE) \not> (ABD) + (ACD) + (ADE) + (BCD) + (BDE)\\ + (CDE) - (AD) - (BD) - (CD) - (DE) + (D)\end{aligned}\right\} \quad [5]$$

$$\left.\begin{aligned}-(ABCD) + (ABCE) + (ABDE) + (ACDE) + (BCDE) \not> (ABE) + (ACE) + (ADE) + (BCE) + (BDE)\\ + (CDE) - (AE) - (BE) - (CE) - (DE) + (E)\end{aligned}\right\} \quad [6]$$

$$-(ABCD) - (ABCE) + (ABDE) + (ACDE) + (BCDE) \not< -(ABC) + (ADE) + (BDE) + (CDE) - (DE) \quad [7]$$

$$-(ABCD) + (ABCE) - (ABDE) + (ACDE) + (BCDE) \not< -(ABD) + (ACE) + (BCE) + (CDE) - (CE) \quad [8]$$

$$-(ABCD) + (ABCE) + (ABDE) - (ACDE) + (BCDE) \not< (ABE) - (ACD) + (BCE) + (BDE) - (BE) \quad [9]$$

$$-(ABCD) + (ABCE) + (ABDE) + (ACDE) - (BCDE) \not< (ABE) + (ACE) + (ADE) - (BCD) - (AE) \quad [10]$$

$$(ABCD) - (ABCE) - (ABDE) + (ACDE) + (BCDE) \not< -(ABE) + (ACD) + (BCD) + (CDE) - (CD) \quad [11]$$

$$(ABCD) - (ABCE) + (ABDE) - (ACDE) + (BCDE) \not< (ABD) - (ACE) + (BCD) + (BDE) - (BD) \quad [12]$$

$$(ABCD) - (ABCE) + (ABDE) + (ACDE) - (BCDE) \not< (ABD) + (ACD) + (ADE) - (BCE) - (AD) \quad [13]$$

$$(ABCD) + (ABCE) - (ABDE) - (ACDE) + (BCDE) \not< (ABC) - (ADE) + (BCD) + (BCE) - (BC) \quad [14]$$

$$(ABCD) + (ABCE) - (ABDE) + (ACDE) - (BCDE) \not< (ABC) + (ACD) + (ACE) - (BDE) - (AC) \quad [15]$$

$$(ABCD) + (ABCE) + (ABDE) - (ACDE) - (BCDE) \not< (ABC) + (ABD) + (ABE) - (CDE) - (AB) \quad [16]$$

§ 10. The conditions of superior congruence given in VII. only become impossible if either of the minor limits for any one set of fourth-order frequencies be greater than either of the major limits (*e.g.*, [3] or [4] greater than [1] or [2]). But on expressing the condition that each minor limit must be less than the major, we see we are simply led back to limits of the form V., § 7. That is to say—"A congruence of the fifth degree and third order is self-consistent, if each of the five congruences of the fourth degree, into which it can be resolved, is self-consistent." But we have seen that comparison of the minor and major limits of conditions V. again leads back simply to conditions III. Therefore we must have also—"A congruence of the fifth degree and second order is self-consistent, if each of the ten congruences of the third degree, into which it can be resolved, is self-consistent." The conditions of consistence for the congruence of any degree are then, so far as we have gone at all events, thrown back on the conditions for the simple congruence, the degree of which only exceeds its order by unity.

Remarks on the Preceding Sections.

General Solution.

§ 11. The elementary method employed in the preceding sections is the one best adapted for exhibiting clearly the physical meaning of the conditions of consistence. It is perfectly adapted for finding the conditions for a congruence of any degree, though the number of comparisons of limits to be made appears at first sight to make the work extremely lengthy. A few considerations, however, rapidly reduce the number of necessary comparisons. Thus all comparisons of expansions due to two groups that are contrary to each other in one term only give *inferior* conditions.

Again, all comparisons of expansions due to groups that are contrary to each other in three terms, give conditions simply derivable from those of the congruence of the third degree. Take for instance the first condition of § 5, III.

$$(AB) + (AC) + (BC) \nless (A) + (B) + (C) - (U).$$

The universe in which this inequality is to hold good is not specified at all. Let it be a universe in which all things are D. Then the condition becomes

$$(ABD) + (ACD) + (BCD) \nless (AD) + (BD) + (CD) - (D).$$

But this is simply the second condition of superior consistence for a congruence of the fourth degree (V., § 7). Again let the universe be not D, but δ. Then the condition becomes

$$(AB\delta) + (AC\delta) + (BC\delta) \nless (A\delta) + (B\delta) + (C\delta) - (\delta),$$

getting rid of the negative terms by expansion, that is

$$(ABD) + (ACD) + (BCD) \ngtr (AB) + (AC) + AD) + (BC) + (BD) + (CD) \\ - (A) - (B) - (C) - (D) + (U).$$

But this is again simply the first condition of V., § 7. The whole of the conditions V. for the congruence of fourth degree may in fact be derived by writing down the conditions III. for each aggregate ABC, ABD, ACD, BCD, inserting the universes D or δ in the first case, C or γ in the second, and so on. Evidently this will give the right number of conditions, there being four aggregates and four conditions for each, while each condition must hold good in two universes, total $4 \times 4 \times 2 = 32$.

A precisely similar theorem holds good for the 160 conditions of § 9, VII., for the congruence of the fifth degree. The congruence of second order and fifth degree may be resolved into *ten* congruences of the third degree. The *four* conditions of III. must hold good for each of these congruences in *four* universes, *e.g.*, for the aggregate ABC, the universes DE, Dϵ, δE, $\delta\epsilon$. The whole number of conditions so derived is $10 \times 4 \times 4$, or 160. The sixteen conditions of VIII., § 9, cannot be so derived; they involve five, not three, fifth-order frequencies each, and are quite new conditions derived from the comparison of expansions of groups contrary to each other in five attributes.

The results suggest, however, that to obtain the *new* conditions for a congruence of degree $2m + 1$, order $2m$, we have only to consider the 2^{2m} possible comparisons of contraries. A congruence of even degree, say $2m$, is subject to no conditions beyond those immediately derivable from the congruence of degree $2m - 1$.

But this result is only suggested, not proved, by the few cases taken; nor are the general theorems corresponding to those given at the end of § 8 and § 10 demonstrable from the mere particular cases.

§ 12. By slightly changing the point of view, and remembering that any frequency may be expanded by considering A, B, C, U, &c., as "elective operators," subject to the ordinary laws of multiplication, and to the special laws

$$U.A = A$$
$$A^n = A,$$

the conditions of congruence may be obtained in a simple general form.

Referring back to the earlier sections (§ 5, § 7, or § 9), it will be seen that in considering the congruence of degree m order $\overline{m-1}$, all the groups of the mth order, containing an even number of negative terms (or attributes), were first taken and expanded, and the expansion put not less than zero, thus giving a system of minor limits for (ABCD . . M). The remaining groups, containing an odd number of negatives, were similarly expanded, giving a system of major limits. The expansions in the two cases were of the forms

$$+ (\text{ABCD} \ldots \text{M}) - (\text{terms of lower order than } m)_1$$
$$- (\text{ABCD} \ldots \text{M}) + (\text{terms of lower order than } m)_2.$$

We took this in the previous sections as giving

$$(\text{ABCD} \ldots \text{M}) \not< (\text{terms of lower order than } m)_1$$
$$(\text{ABCD} \ldots \text{M}) \not> (\text{terms of lower order than } m)_2,$$

and deduced

$$(\text{terms of lower order than } m)_2 \not< (\text{terms of lower order than } m)_1.$$

But precisely the same result is arrived at by adding the two expansions and putting the sum not less than zero,* and this is a much more convenient conception from which to obtain the general conditions.

Let the two groups contain positive attributes $K_1K_2 \ldots K_p$ and negative attributes $\lambda_1\lambda_2 \ldots \lambda_q$ in common; positive attributes $M_1M_2 \ldots M_r$ in the one, with their negatives $\mu_1\mu_2 \ldots \mu_r$ in the other; and negative attributes $\nu_1\nu_2 \ldots \nu_s$ in the one, with their positives $N_1N_2 \ldots N_s$ in the other. Using the symbol Π to denote "the continued operator-product of all quantities like . . ." the expansions of the two frequencies may be written

$$\prod_{p=1}^{p=p}[\text{K}_p] \prod_{q=1}^{q=q}[\text{U} - \text{L}_q] \prod_{r=1}^{r=r}[\text{M}_r] \prod_{s=1}^{s=s}[\text{U} - \text{N}_s]$$
$$\prod_{p=1}^{p=p}[\text{K}_p] \prod_{q=1}^{q=q}[\text{U} - \text{L}_q] \prod_{r=1}^{r=r}[\text{U} - \text{M}_r] \prod_{s=1}^{s=s}[\text{N}_s].$$

In these expressions it must be remembered that as the one group is to contain an even, the other an odd, number of negatives, if $q + r$ be odd $q + s$ must be even, and *vice versâ*. Hence $r + s$ must in any case be odd, *i.e.*, the two groups must be contrary as regards an odd number of attributes, for if

$$q + s = 2x$$
$$q + r = 2y + 1$$
$$r + s = 2(x + y - q) + 1,$$

which is necessarily odd. The general condition of consistence for a congruence of degree

$$m = p + q + r + s,$$

* Every frequency must be greater than zero, and *à fortiori* the sum of any two. But it is only by taking the sum of two frequencies, the one containing an even, and the other an odd, number of negatives that an expression is obtained from which the mth-order term is eliminated.

and order $(m - 1)$ may then be written

$$\overset{p}{\underset{1}{\Pi}}[K_p] \quad \overset{q}{\underset{1}{\Pi}}[U - L_q]\left\{\overset{r}{\underset{1}{\Pi}}[M_r] \quad \overset{s}{\underset{1}{\Pi}}[U - N_s] + \overset{r}{\underset{1}{\Pi}}[U - M_r] \quad \overset{s}{\underset{1}{\Pi}}[N_s]\right\} \not< 0 \quad . \quad (1).$$

All the conditions of consistence, whether inferior or superior, given in the preceding sections may be readily verified from this general expression.*

§ 13. If the two groups compared be contrary in c terms $(c = r + s)$, the expansions will give rise to c terms of the $(m - 1)$th order, viz., s from the first term, and r from the second term within the curly bracket. Thus the conditions VII., § 9, with *three* fourth-order frequencies on the left, were all obtained by comparison of expansions of fifth-order frequencies contrary in *three* attributes; the conditions VIII. on the other hand by comparison of expansions due to fifth-order frequencies contrary in all five attributes.

The term outside the bracket in (1) may be regarded as a mere specification of the universe within which the simple condition

$$\overset{r}{\underset{1}{\Pi}}[M_r] \quad \overset{s}{\underset{1}{\Pi}}[U - N_s] + \overset{r}{\underset{1}{\Pi}}[U - M_r] \quad \overset{s}{\underset{1}{\Pi}}[N_s] \not< 0 \quad . \quad . \quad . \quad . \quad . \quad (2),$$

is to hold good. For consider the conditions of the general form (1) in which the contrary terms (those within the bracket) are the same, but in which the universe-terms outside the bracket are contrary as regards one attribute, say K_p. Then the universes are specified by

$$K_p \, . \overset{p-1}{\underset{1}{\Pi}}[K_p] \, . \, \overset{q}{\underset{1}{\Pi}}[U - L_q].$$

$$[U - K_p] \overset{p-1}{\underset{1}{\Pi}}[K_p] \, . \, \overset{q}{\underset{1}{\Pi}}[U - L_q],$$

and if the corresponding conditions (1) be added, the term in K_p goes out, leaving a condition of one degree lower. By addition of successive pairs of conditions in this way it is evident that the universe-terms may be entirely eliminated, and only condition (2) left. By the converse process of *specification* of the universe the conditions involving c terms of the $(m - 1)$th order may *always* be obtained from the conditions for the congruence of the cth degree and $(c - 1)$th order, a property on which we have remarked while considering the congruences of low orders investigated in § 4—§ 10; the conditions (2) merely require to be specified for all possible universes.

If instead of proceeding to the entire elimination of the universe-terms we stop short at conditions of degree n $(n \not< c)$, the whole series of conditions so obtained may be grouped into sets arranged according to the attributes that have *not* been

* They were actually first obtained by the method there described.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

eliminated, one such set corresponding to each possible congruence of the nth degree and $(n-1)$th order. Hence the general theorem—"A congruence of the $(n-1)$th order and mth degree is self-consistent if all the possible congruences of the nth degree, into which it may be resolved, are self-consistent." This is the generalisation of the theorems given in § 8 and § 10.

§ 14. The number of conditions to which any congruence of the mth degree is subject is readily obtained. Consider first the congruence of order $(m-1)$. The number of combinations of c (positive) attributes that can be selected from m is

$$\frac{m(m-1)\ \ldots\ (m-c+1)}{(c)!}.$$

From each of these combinations can be generated 2^{c-1} contrary pairs, by negativing one or more of the attributes: *e.g.*, from (ABC) can be formed the *four* contrary pairs

$$\left.\begin{matrix}\mathrm{ABC}\\ \alpha\beta\gamma\end{matrix}\right\} \quad \left.\begin{matrix}\alpha\mathrm{BC}\\ \mathrm{A}\beta\gamma\end{matrix}\right\} \quad \left.\begin{matrix}\mathrm{A}\beta\mathrm{C}\\ \alpha\mathrm{B}\gamma\end{matrix}\right\} \quad \left.\begin{matrix}\mathrm{AB}\gamma\\ \alpha\beta\mathrm{C}\end{matrix}\right\}.$$

From the remaining $(p+q)$ or $(m-c)$ attributes can be formed 2^{m-c} different universes. Hence the number of conditions involving c terms of the $(m-1)$th order is

$$2^{m-1}\frac{m(m-1)\ \ldots\ (m-c+1)}{(c)!}.$$

The *whole* number of such conditions (including those of inferior congruence) will be given by inserting all possible values of c in the above expression (viz., all odd numbers not greater than m), and summing. That is

$$2^{m-1}\left(m+\frac{m(m-1)(m-2)}{1.2.3}+\frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{1.2.3.4.5}+\ \ldots\ \right),$$

or, substituting $1+\overline{m-1}$ for m and rearranging

$$2^{m-1}\left(1+(m-1)+\frac{(m-1)(m-2)}{1.2}+\frac{(m-1)(m-2)(m-3)}{1.2.3}+\ \ldots\ \right)=2^{2(m-1)}.$$

The number of conditions is thus quadrupled for every unit rise in the degree of the congruence, and grows with extreme rapidity. The actual numbers, up to the congruence of ninth degree and eighth order are given in Table I. below. For more than five or six attributes the actual arithmetic discussion of any particular case seems to pass the bounds of practical possibilities.

TABLE I.—Number of Conditions of Consistence for a Congruence of the mth degree and $(m-1)$th order; specifying separately the number involving 3, 5, 7, or 9 $(m-1)$th order terms.

Terms involved $(= c)$.	Degree of congruence $= m$.						
	3.	4.	5.	6.	7.	8.	9.
Inferior congruence, 1	12	32	80	192	448	1024	2304
3	4	32	160	640	2240	7168	21504
5	—	—	16	192	1344	7168	32256
7	—	—	—	—	64	1024	9216
9	—	—	—	—	—	—	256
Total superior congruence only .	4	32	176	832	3648	15360	63232
Grand total	16	64	256	1024	4096	16384	65536

The whole number of conditions for a congruence of the nth order and mth degree is

$$2^{2n} \cdot \frac{m(m-1)(m-2)\ldots(m-n)}{(n+1)!},$$

viz., the number of conditions for a congruence of order n, degree $(n+1)$, multiplied by the number of $(n+1)$th degree congruences into which the mth-degree congruence can be resolved. The actual figures are—

TABLE II.—Whole Number of Conditions of Consistence for a Congruence of degree m order n.

Order of congruence $= n$.	Degree of congruence $= m$.						
	3.	4.	5.	6.	7.	8.	9.
2	16	64	160	320	560	896	1344
3		64	320	960	2240	4480	8064
4			256	1536	5376	14336	32256
5				1024	7168	28672	86016
6					4096	32768	147456
7						16384	147456
8							65536

Geometrical Representation of the Conditions of Consistence.

Congruence of the Third Degree.

§ 15. Since the conditions of consistence for a congruence of the third degree only involve three second-order frequencies each, it is possible to construct geometrical models to represent them, the first-order terms being treated as constants. These models exhibit in such a beautiful manner the nature of the conditions, and the limiting character of the cases dealt with in ordinary logic, that it is worth while to treat a few special cases at length as illustrations. It will be convenient to use for the present the abbreviated notation

$$x = (\mathrm{AB})/(\mathrm{U}) \;.\; y = (\mathrm{AC})/(\mathrm{U}) \;.\; z = (\mathrm{BC})/(\mathrm{U}).$$
$$p_1 = (\mathrm{A})/(\mathrm{U}) \;.\; p_2 = (\mathrm{B})/(\mathrm{U}) \;.\; p_3 = (\mathrm{C})/(\mathrm{U}).$$

Then, treating x, y, z as rectangular co-ordinates, all sets of consistent values ot x, y, z must determine points within the space bounded by planes (§ 4 and § 5).

$$\left.\begin{array}{llll} x = 0 & \text{or } x = p_1 + p_2 - 1, & x = p_1 & \text{or } x = p_2 \\ y = 0 & \;,,\; y = p_1 + p_3 - 1, & y = p_1 & \;,,\; y = p_3 \\ z = 0 & \;,,\; z = p_2 + p_3 - 1, & z = p_2 & \;,,\; z = p_3 \end{array}\right\} \quad . \quad . \quad . \quad (\mathrm{A}).$$

$$\left.\begin{array}{ll} x + y + z = p_1 + p_2 + p_3 - 1 & (\alpha) \\ x + y - z = p_1 & (\beta) \\ x - y + z = p_2 & (\gamma) \\ -x + y + z = p_3 & (\delta) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B}).$$

It is convenient to regard the equilateral tetrahedron bounded by the four planes (α) — (δ), representing the superior conditions of consistence, as the fundamental "congruence surface," its edges being truncated more or less by the planes (A). Only six of the planes (A) *at most* can of course come into account at one time, the remaining six lying outside the surface.

The lines $(\alpha\beta)$, $(\alpha\gamma)$, $(\alpha\delta)$, &c., in which the planes α, β, γ, δ meet, are all parallel to one or other of the co-ordinate planes: thus we have for

$$\left.\begin{array}{ll} (\alpha\beta) & z = \tfrac{1}{2}\,(p_2 + p_3 - 1) \\ (\alpha\gamma) & y = \tfrac{1}{2}\,(p_1 + p_3 - 1) \\ (\alpha\delta) & x = \tfrac{1}{2}\,(p_1 + p_2 - 1) \\ (\beta\gamma) & x = \tfrac{1}{2}\,(p_1 + p_2) \\ (\beta\delta) & y = \tfrac{1}{2}\,(p_1 + p_3) \\ (\gamma\delta) & z = \tfrac{1}{2}\,(p_2 + p_3) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{C}).$$

The plan and elevation of the *complete* tetrahedron are thus both squares of side $= 0{\cdot}5$. But, comparing equations (C) with (A), the edge

$$
\begin{array}{llll}
(\alpha\beta) & \text{is truncated unless} & & p_2 + p_3 = 1. \\
(\alpha\gamma) & \quad ,, & ,, & p_1 + p_3 = 1. \\
(\alpha\delta) & \quad ,, & ,, & p_1 + p_2 = 1. \\
(\beta\gamma) & \quad ,, & ,, & p_1 = p_2. \\
(\beta\delta) & \quad ,, & ,, & p_1 = p_3. \\
(\delta\gamma) & \quad ,, & ,, & p_2 = p_3,
\end{array}
$$

§ 16. Case (1).

$$p_1 = p_2 = p_3 = 0{\cdot}5.$$

In this case the congruence-surface reduces to the fundamental tetrahedron, fig. 1*a*,* the planes (A) only passing through its edges and not truncating them. The form is more clearly shown by fig. 1*b*, which is drawn from a photograph of an actual

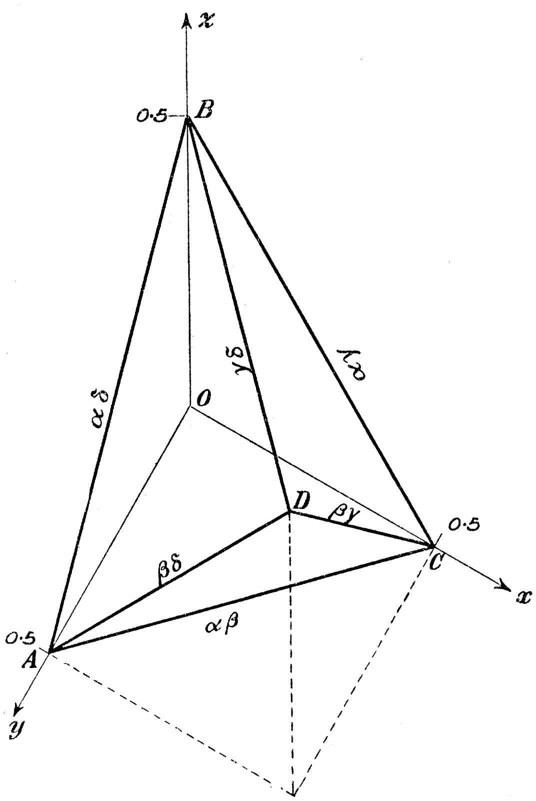


Fig. 1*a*.

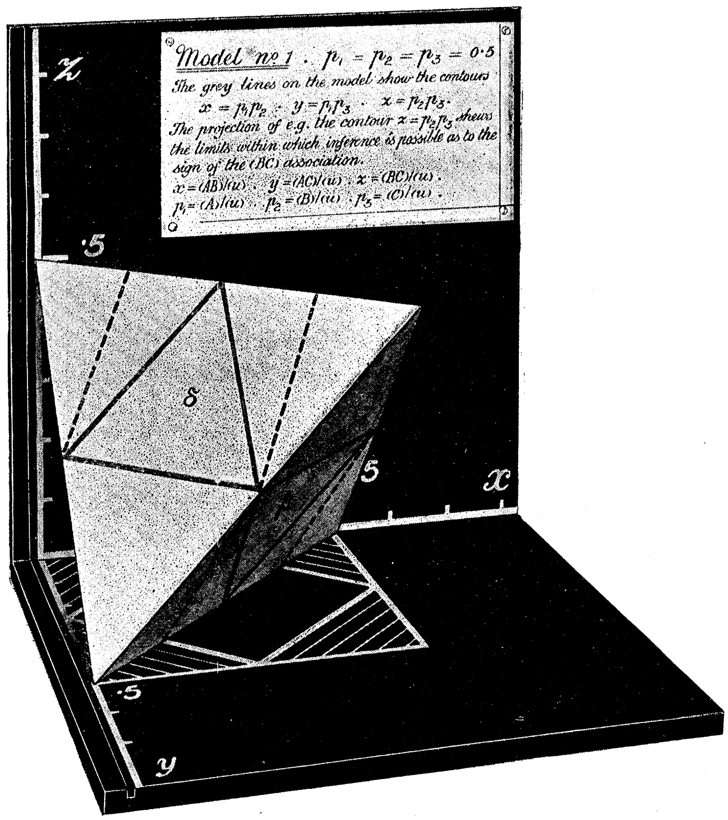


Fig. 1*b*.

* This and the similar following figures are drawn in orthographic projection. The picture-plane is parallel to the axis of z, and its trace on the plane of xy makes an angle of 30° with the x axis. The generators lie in planes perpendicular to the picture-plane and the plane of xy, but make an angle of 45 with the latter plane. The observer must therefore imagine himself to be looking *down* on the model.

model. The contours shown in this figure and in the subsequent figs. 2*b*, 4*b*, and 4*c*, do not at present concern us.

The bounding planes are

$$x + y + z = 0\cdot5 \qquad (\alpha)$$
$$x + y - z = 0\cdot5 \qquad (\beta)$$
$$x - y + z = 0\cdot5 \qquad (\gamma)$$
$$-x + y + z = 0\cdot5 \qquad (\delta).$$

If the ordinate z corresponding to given values of x and y be drawn, it will in general cut the surface in two points. These determine the upper and lower limits to values of z consistent with the given values of x and y. If however x and y determine a point on the plan of one of the edges of the tetrahedron, z only cuts the surface in one point, and its value can be inferred. Thus we have the following cases of logical inference :—

Given.	Inferred.
$x = 0$	$y = 0\cdot5 - z$
$x = 0\cdot5$	$y = z$
$y = 0$	$z = 0\cdot5 - x$
$y = 0\cdot5$	$z = x$
$z = 0$	$x = 0\cdot5 - y$
$z = 0\cdot5$	$x = y$.

Each of these cases corresponds, it will be noticed, to a single-infinity of special inferences from two data—one to every point on each edge. It is the only instance in which six such infinite series of possible inferences occur, that is, six series of *exact* inferences—inferences of a "universal affirmative" or "universal negative," to use the logical terms.

§ 17. The limits to x, y, or z given for this case by equations (α)-(δ) at the beginning of last section, are precisely those deducible from quite different considerations for the quadrant frequencies (above and below average) in the case of normal correlation. In that case, if r_{12}, r_{13}, r_{23} be the three correlation coefficients, we have for the limits to r_{23}*

$$r_{12}r_{13} \pm \sqrt{1 + r_{12}^2 r_{13}^2 - r_{12}^2 - r_{13}^2}.$$

But by the theorem due to Mr. W. F. SHEPPARD†

$$r_{12} = - \cos 2x\pi$$
$$r_{13} = - \cos 2y\pi$$
$$r_{23} = - \cos 2z\pi.$$

* 'Roy. Soc. Proc.,' 1897, vol. 60, p. 486. In the first line of the table on p. 486, for "0" on the right *read* ± 1. A similar correction is to be made in the 'Journal of the Roy. Stat. Soc.,' vol. 60, p. 834.

† 'Phil. Trans.,' A, 1898, vol. 162, p. 101.

Therefore the limits to $-\cos 2z\pi$ are

$$\begin{aligned}&\cos 2x\pi \,.\, \cos 2y\pi \pm \sqrt{1 + \cos^2 2x\pi \cos^2 2y\pi - \cos^2 2x\pi - \cos^2 2y\pi} \\ &\quad = \cos 2x\pi \,.\, \cos 2y\pi \pm \sin 2x\pi \sin 2y\pi \\ &\quad = \cos (x \pm y) 2\pi.\end{aligned}$$

Also

$$-\cos 2z\pi = \cos (z \pm \tfrac{1}{2}) 2\pi.$$

Therefore the limiting values to z are given by

$$\pm \tfrac{1}{2} \pm (x \pm y).$$

Here we need not take all three signs positive, for by the inferior conditions z cannot be greater than 0·5; nor all three negative, for x cannot be less than zero. Hence the limits given are

$$\left.\begin{aligned} 0{\cdot}5 + x - y \\ 0{\cdot}5 - x + y \end{aligned}\right\}$$
$$\left.\begin{aligned} 0{\cdot}5 - x - y \\ -0{\cdot}5 + x + y \end{aligned}\right\}$$

These are precisely the limits given by the conditions of consistence stated at the commencement of § 16. Thus the limits to one correlation coefficient in terms of the two others are most simply regarded as functions of the limits to the quadrant frequencies. The table below gives the limits to z for different values of x and y; the table is of course symmetrical with regard to x and y.

Table showing the limits to z in terms of x and y, $x = (AB)/(U)$, $y = (AC)/(U)$, $z = (BC)/(U)$, for the case of Equality of Contraries.

Value of y.	Value of x.					
	0.	0·1.	0·2.	0·3.	0·4.	0·5.
0	0·5	0·4	0·3	0·2	0·1	0
0·1	0·4	0·5 0·3	0·4 0·2	0·3 0·1	0·2 0	0·1
0·2	0·3	0·4 0·2	0·5 0·1	0·4 0	0·3 0·1	0·2
0·3	0·2	0·3 0·1	0·4 0	0·5 0·1	0·4 0·2	0·3
0·4	0·1	0·2 0	0·3 0·1	0·4 0·2	0·5 0·3	0·4
0·5	0	0·1	0·2	0·3	0·4	0·5

§ 18. Case (2).

$$p_1 = p_2 = p_3 = 0{\cdot}4.$$

The equations to the bounding planes are

$$x = 0 \qquad y = 0 \qquad z = 0.$$
$$x + y + z = 0{\cdot}2 \qquad (\alpha)$$
$$x + y - z = 0{\cdot}4 \qquad (\beta)$$
$$x - y + z = 0{\cdot}4 \qquad (\gamma)$$
$$-x + y + z = 0{\cdot}4 \qquad (\delta).$$

The surface is shown in fig. $2a$ below, and in fig. $2b$ from a photograph of a model.

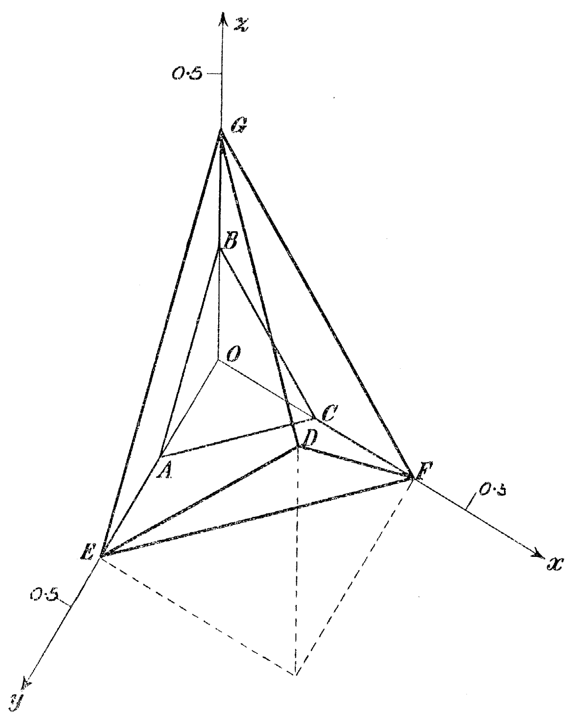


Fig. $2a$.

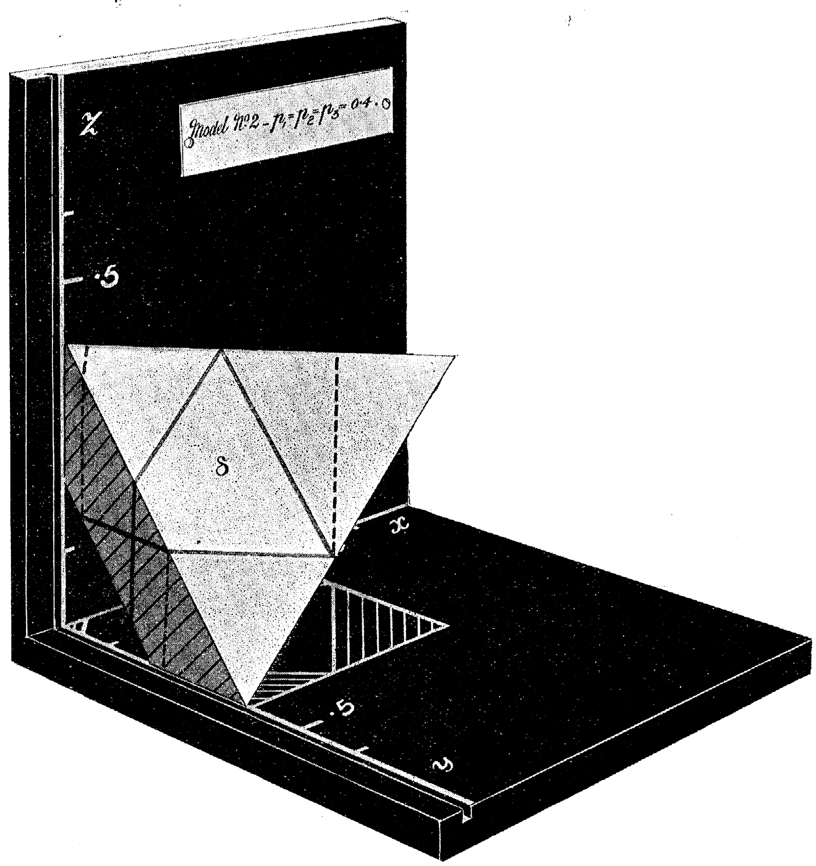


Fig. $2b$.

The three edges $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$ of the tetrahedron, that in fig. 1 lay in the co-ordinate planes, are now truncated by them. Thus only three of the six infinite series of exact inferences occurring in the last case are left, viz.,

Given.	Inferred.
$x = 0{\cdot}4$	$y = z$
$y = 0{\cdot}4$	$z = x$
$z = 0{\cdot}4$	$x = y.$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

It should be noticed that from $x = 0$ $y = 0$ we can *in the present case*, infer $z > 0{\cdot}2$, *i.e.*, from "no A's are B" and "no A's are C" infer "some (at least half) of the B's are C." But the reason why the ordinary rule of logic, "from two negative premises no conclusion can be drawn," comes into play is, however, fairly obvious. If

$$p_1 + p_2 + p_3 < 1$$

the plane α disappears behind the origin of co-ordinates, and the rear of the surface is bounded solely by the co-ordinate planes. The value of z may then be anywhere between zero and p_1 or p_2 at the point $x = 0$ $y = 0$, *i.e.*, there is, *à priori*, no inference.

I do not propose to enter into the discussion of numerically indefinite inferences—*i.e.*, the "particular affirmative" and "particular negative" conclusions of the ordinary syllogistic treatment. I would, however, suggest that such numerically indefinite inferences may be regarded as mere degradations, owing to the truncation of the tetrahedron, of the series of exact or definite inferences possible in the last case.

§ 19. Case (3).

$$p_1 = 0{\cdot}45 \qquad p_2 = p_3 = 0{\cdot}4.$$

The equations to the bounding planes are

$$x = 0 \qquad y = 0 \qquad z = 0$$

$$x = 0{\cdot}4 \qquad y = 0{\cdot}4 \qquad z\ 0{\cdot}4$$

$$x + y + z = 0{\cdot}25 \qquad (\alpha)$$

$$x + y - z = 0{\cdot}45 \qquad (\beta)$$

$$x - y + z = 0{\cdot}4 \qquad (\gamma)$$

$$-x + y + z = 0{\cdot}4 \qquad (\delta).$$

The form of the surface is shown in fig. 3. Only one edge of the primitive tetrahedron is now untruncated, viz., GK or $\gamma\delta$. Hence only one of the original six series of definite inferences is left, viz.—

Given.	Inferred.
$z = 0{\cdot}4$	$x = y.$

In addition to this infinite series there are, however, two special cases of inference corresponding respectively to the points H and M of the figure.

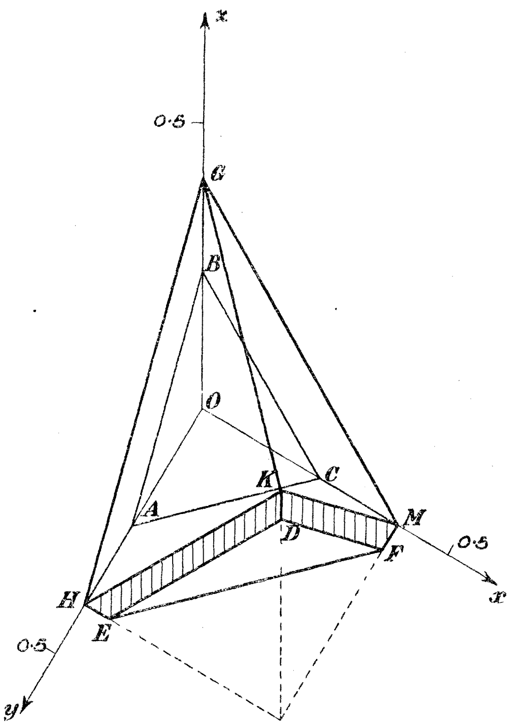


Fig. 3.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Given.	Inferred.
$x = 0 \quad . \; y = 0{\cdot}4$	$z = 0$
$x = 0{\cdot}4 \; . \; y = 0$	$z = 0.$

These are "universal-negative" conclusions of the ordinary syllogistic type

All C's are A, No A's are B, ∴ No C's are B.	All B's are A, No A's are C, ∴ No B's are C.

§ 20. Case (4).

A type of the most general case possible.

$$p_1 = 0{\cdot}35 \; : \; p_2 = 0{\cdot}4 \; : \; p_3 = 0{\cdot}45.$$

The equations to the bounding planes are

$$x = 0 \qquad y = 0 \qquad z = 0$$
$$x = 0{\cdot}35 \quad y = 0{\cdot}35 \quad z = 0{\cdot}4$$
$$x + y + z = 0{\cdot}20 \qquad (\alpha)$$
$$x + y - z = 0{\cdot}35 \qquad (\beta)$$
$$x - y + z = 0{\cdot}40 \qquad (\gamma)$$
$$- x + y + z = 0{\cdot}45 \qquad (\delta).$$

The form of the surface is shown in fig. 4*a* opposite, and in figs. 4*b* and 4*c* from photographs of a model. All the edges of the tetrahedron are now truncated by planes representing the conditions of inferior congruence. No infinite series of definite inferences are left, but only four special cases corresponding to the points KEGF of the figure :—

	Given.	Inferred.
(1)	$x = 0{\cdot}35 \; . \; z = 0{\cdot}4$	$y = 0{\cdot}35$
(2)	$y = 0{\cdot}35 \quad z = 0$	$x = 0$
(3)	$y = 0 \quad z = 0{\cdot}4$	$x = 0$
(4)	$x = 0{\cdot}35 \quad z = 0$	$y = 0.$

The corresponding syllogisms are—

(1.) All A's are B, all B's are C, ∴ all A's are C.	(3.) All B's are C, no C's are A, ∴ no B's are A.
(2.) All A's are C, no C's are B, ∴ no A's are B.	(4.) All A's are B, no B's are C, ∴ no A's are C.

§ 21. In extreme cases the congruence-surface presents the appearance of a right six-face with its corners truncated rather than a tetrahedron with its edges cut down.

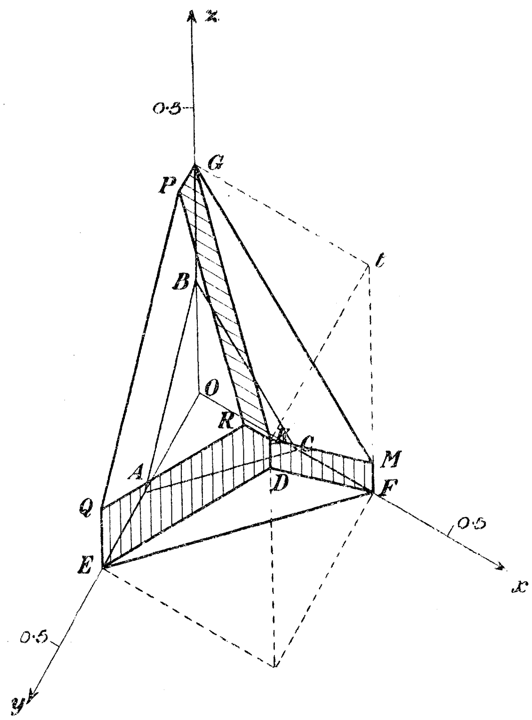


Fig. 4*a*.

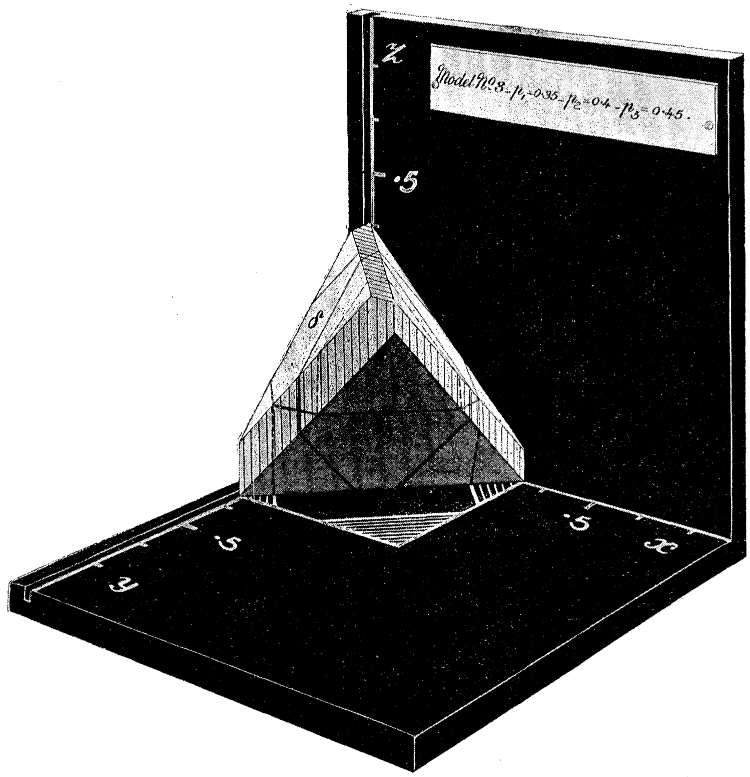


Fig. 4*b*.

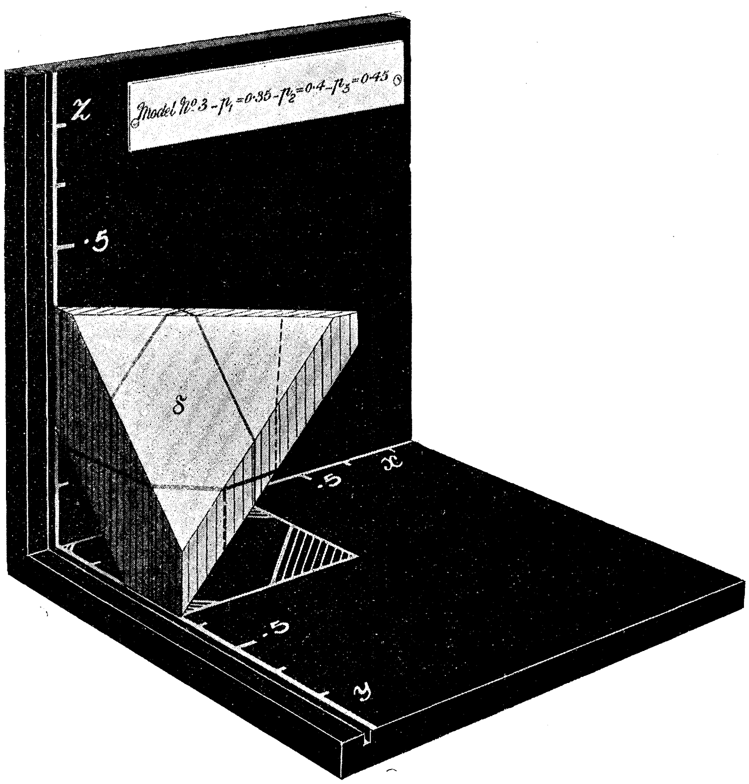


Fig. 4*c*.

Illustrations of such surfaces will be found in figs. 16–23 (§ 36, p. 130–131). The inferior instead of the superior conditions then predominate in importance.

Two of the planes of superior congruence—but not more than two—may simultaneously pass outside the planes of inferior congruence and so disappear. Thus to determine the planes of inferior congruence suppose

$$p_1 < p_2 < p_3 \quad \text{and} \quad p_1 < 0{\cdot}5,\ p_2 < 0{\cdot}5,\ p_3 < 0{\cdot}5.$$

The plane α vanishes if

$$p_1 + p_2 + p_3 < 1.$$

The plane β cuts the plane $z = 0$ in the line

$$x + y = p_1;$$

the limiting values of x and y being $x = p_1$, $y = p_1$ it cannot disappear.

The plane γ cuts the plane $z = p_2$ (the limiting plane) in the line

$$x - y = 0,$$

and therefore it also cannot disappear.

Finally, the plane δ cuts the plane $z = p_2$ in the line

$$y - x = p_3 - p_2.$$

But the greatest possible value of $y - x$ is p_1. Therefore the plane δ disappears if

$$p_3 - p_2 > p_1.$$

An illustration of a surface of this character will be found in fig 19, § 36, p. 130.

In any case there are four "syllogism points" (like KEGF of fig. 4) to the surface.

§ 22. In all the preceding examples of figs. (1)–(4) the values assigned to p_1, p_2, and p_3 have been less than 0·5, so that the edges $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$ of the primitive tetrahedron were truncated, if truncated at all, by the co-ordinate planes. If

$$\begin{aligned} p_1 + p_2 &> 1 \\ p_2 + p_3 &> 1 \\ p_3 + p_1 &> 1 \end{aligned}$$

this ceases to be the case (*vide* equations A § 15), the truncating planes then move inwards, and the congruence-surface stands clear of the co-ordinate planes.

A little consideration will show, however, that no new features are introduced into the surface itself. Thus suppose that p_1, p_2, p_3 are all less than 0·5, but that we then substitute $(1-p_3)$ for p_3 so as to make one ratio greater than 0·5. How is the original surface altered? Substituting $(1-p_3)$ for p_3 amounts to substituting γ for C. Then if x_1, y_1, z_1 be the original co-ordinates, x_2, y_2, z_2 the co-ordinates after the substitution, we must have

$$x_2 = x_1$$
$$y_2 = p_1 - y_1$$
$$z_2 = p_2 - z_1,$$

since

$$(A\gamma) = (A) - (AC). \qquad (B\gamma) = (B) - (BC).$$

The first surface is therefore changed into the second by a simple transformation of co-ordinates, *i.e.*, if we are dealing with an actual model of the surface, by turning it over and shifting it.

Fig. 5 is drawn to illustrate the nature of an actual transformation. It is drawn for the values

$$p_1 = 0{\cdot}35 \quad p_2 = 0{\cdot}4 \quad p_3 = 0{\cdot}55,$$

substituting $(1-0{\cdot}45)$ for the $0{\cdot}45$ assigned to p_3 in fig. $4a$, p. 115. The two figures are similarly lettered. The model of fig. 4 has been turned over, round an axis parallel to the axis of x, through a half revolution.

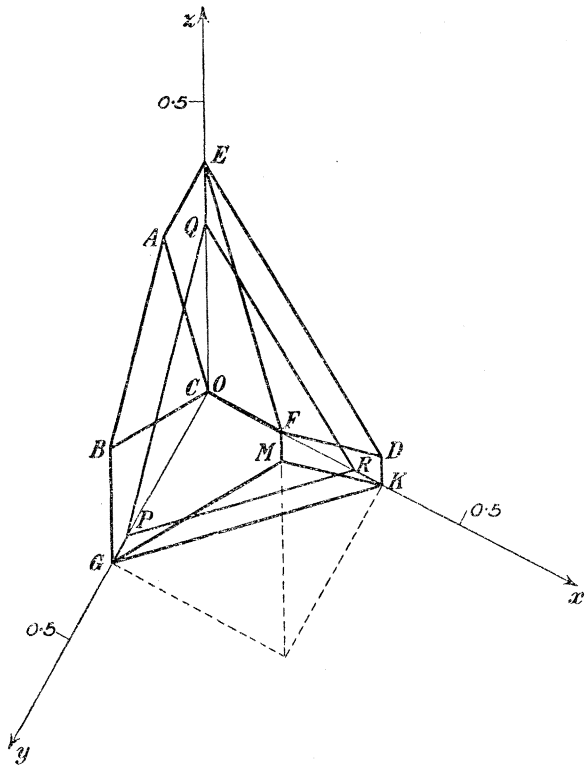


Fig. 5.

§ 23. If $(1-p_3)$, $(1-p_2)$, $(1-p_1)$ be *successively* substituted for p_3, p_2, p_1 the transformations are as follows:—

(1.) Substituting $(1-p_3)$ for p_3

$$x_2 = x_1$$
$$y_2 = p_1 - y_1$$
$$z_2 = p_2 - z_1$$

(2.) Substituting $(1-p_2)$ for p_2

$$\begin{aligned} x_3 &= p_1 - x_2 &&= p_1 - x_1 \\ y_3 &= y_2 &&= p_1 - y_1 \\ z_3 &= 1 - p_3 - z_2 &&= 1 - p_2 - p_3 + z_1. \end{aligned}$$

(3.) Substituting $(1-p_1)$ for p_1

$$\begin{aligned} x_4 &= 1 - p_2 - x_3 &&= 1 - p_1 - p_2 + x_1 \\ y_4 &= 1 - p_3 - y_3 &&= 1 - p_1 - p_3 + y_1 \\ z_4 &= z_3 &&= 1 - p_2 - p_3 + z_1. \end{aligned}$$

The second and third cases are obtained, like the first, by simply expanding. Thus $z_3 = (\beta\gamma)/(u)$, and

$$\begin{aligned} (\beta\gamma) &= (\gamma) - (B\gamma) = (U) - (C) - (B\gamma) \\ &= (U) - (B) - (C) + (BC) \end{aligned}$$

or, dividing by (U)

$$z_3 = 1 - p_3 - z_2 = 1 - p_2 - p_3 + z_1$$

as above.

§ 24. The correctness of the transformations given may, of course, be verified directly. Thus suppose

$$p_1 < p_2 < p_3 < 0{\cdot}5$$

then the equations to the bounding planes of the congruence-surface are

$$\left.\begin{array}{lll} x = 0 & y = 0 & z = 0 \\ x = p_1 & y = p_1 & z = p_2 \\ \multicolumn{3}{l}{x + y + z = p_1 + p_2 + p_3 - 1} \\ \multicolumn{3}{l}{x + y - z = p_1} \\ \multicolumn{3}{l}{x - y + z = p_2} \\ \multicolumn{3}{l}{-x + y + z = p_3} \end{array}\right\} \quad \ldots\ldots\ldots \quad \text{I.}$$

If $(1-p_2)$ be substituted for p_2, and $(1-p_3)$ for p_3, then

$$p_1 < 1 - p_3 < 1 - p_2,$$

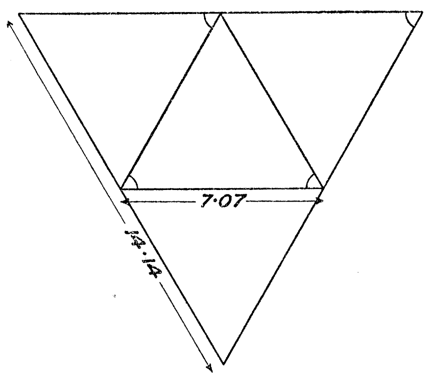
and the equations must be

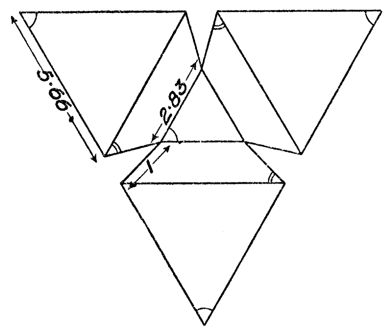
$$\left.\begin{array}{lll} x = 0 & y = 0 & z = 1 - p_2 - p_3 \\ x = p_1 & y = p_1 & z = 1 - p_3 \\ \multicolumn{3}{l}{x + y + z = p_1 - p_2 - p_3 - 1} \\ \multicolumn{3}{l}{x + y - z = p_1} \\ \multicolumn{3}{l}{x - y + z = 1 - p_2} \\ \multicolumn{3}{l}{-x + y + z = 1 - p_3} \end{array}\right\} \quad \ldots\ldots\ldots \quad \text{II.}$$

The set of equations II. may be obtained from the set I. by the transformations (2) above, thus affording the verification.

The fact that the model for *any* values of p_1, p_2, and p_3 can always be transformed into a model for the case $p_1 < 0{\cdot}5$, $p_2 < 0{\cdot}5$, $p_3 < 0{\cdot}5$ justifies our terming case (4), § 20, "a type of the most general case." The geometrical transformations here suggested would seem to correspond to "reductions" of the syllogisms. Thus in fig. 5 the point F stands where the point K stood in fig. 4*a*. But the point F of fig. 4*a* corresponds to a syllogism in *Celarent*, *Cesare*, *Camenes* or *Camestres*; F in fig. 5 to a syllogism in *Barbara*. The transformation of co-ordinates corresponds to a reduction of either of the first four forms to the last.

§ 25. In case any of those who read this memoir should care to construct models of the congruence-surfaces illustrated in figs. 1–4, I give dimensioned sketches of their developments below, figs. 6–9. These developments are on half the scale of the projections shown in the preceding figures. An angle with one arc across it is an angle of 60°, with two arcs 45° or 135°; an angle blocked in is a right angle.





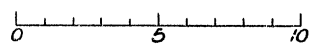
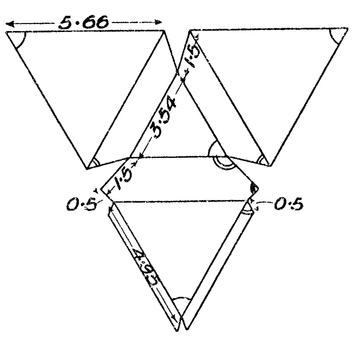


Fig. 6.

Fig. 7.



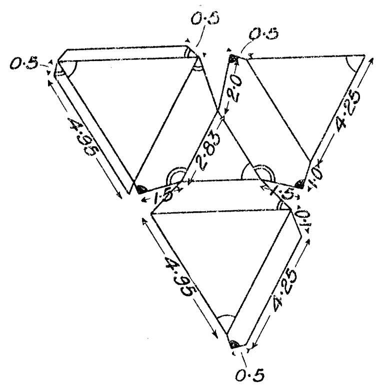


Fig. 8.

Fig. 9.

Congruence of the Fourth Degree.

§ 26. The conditions of consistence for the congruence of the fourth degree in general require space of four dimensions for their direct representation (*cf.* V., § 7).

Since, however, each of the bounding hyper-planes is parallel to one or other of the axes, four surfaces in three dimensions may be substituted for the hyper-surface in four dimensions. If we denote

$$(\mathrm{ABC})/(u), \qquad (\mathrm{ABD})/(u), \qquad (\mathrm{ACD})/(u), \qquad (\mathrm{BCD})/(u),$$

by x, y, z, w, then the surfaces representing the consistence of [1] xzy, [2] xyw, [3] xzw, [4] yzw, respectively, for given values of the frequencies of lower orders, will in general differ from each other. If we desire to find the limits to values of w for given values of x, y, z, we must first see that the values of x, y, z are consistent from surface [1], then find the three pairs of major and minor limits to w given by surfaces [2], [3], and [4]. The lowest of the major limits and highest of the minor limits so given are the true limits to w.

We may take as examples of fourth-degree congruence-surfaces three of the very simplest cases, in which (1) equality of contraries subsists for the first and second-order frequencies, and (2) actual equality subsists between all the second order frequencies, *i.e.*, (AB) = (AC) = (AD) = (BC) = &c. These are highly specialised examples, but are of some interest for their bearing on the theory of normal or quasi-normal correlation.

It must be remembered that in a *normal* distribution of frequency, where the divisions between A and α, B and β, &c., are taken at the means, the frequencies of all orders are definitely determined by the second-order frequencies, and complete equality of contraries subsists for frequencies of all orders. This is not the case where the only datum is that equality of contraries subsists for frequencies of the first and, therefore, also of the second order. The equality of contraries need not, as shown in my previous memoir,* spread to the frequencies of the third order, but if it be assumed to do so the latter become determinate as in the "normal" case.

§ 27. For the simple type of cases considered, the equations to the bounding-planes of the congruence-surfaces reduce to the forms given below. The four component surfaces for xyz, xyw, &c., are, of course, all identical, so it is only necessary to consider one in each case. The first conditions given are those of inferior congruence (*cf.* § 5), and for brevity we have written

$$(\mathrm{AB})/(u) = (\mathrm{AC})/(u) = \&\mathrm{c.} = q$$

$$\begin{array}{lll} x = 0 & y = 0 & z = 0 \\ \;\; = 2q - 0{\cdot}5 & \;\; = 2q - 0{\cdot}5 & \;\; = 2q - 0{\cdot}5 \\ x = q & y = q & \;\; = q \\ \;\; = 3q - 0{\cdot}5 & \;\; = 3q - 0{\cdot}5 & \;\; = 3q - 0{\cdot}5 \end{array}$$

* "On the Association of Attributes in Statistics," 'Phil. Trans.,' A, vol. 194, pp. 263 *et seq.*

$$x + y + z = 3q - 0{\cdot}5 \qquad (\alpha)$$
$$x + y - z = q \qquad (\beta)$$
$$x + y + z = q \qquad (\gamma)$$
$$-x + y + z = q \qquad (\delta)$$
$$x + y + z = 6q - 1{\cdot}0 \qquad (\epsilon)$$
$$x + y - z = 2q - 0{\cdot}5 \qquad (\zeta)$$
$$x + y + z = 2q - 0{\cdot}5 \qquad (\eta)$$
$$-x + y + z = 2q - 0{\cdot}5 \qquad (\theta).$$

§ 28. Case (1)—

$$(AB)(u) = (BC)(u) = \&c. = q = 0{\cdot}25$$

All the attributes ABCD are independent *pair and pair*.

The conditions of inferior congruence are

$$x = 0 \qquad y = 0 \qquad z = 0$$
$$x = 0{\cdot}25 \quad y = 0{\cdot}25 \quad z = 0{\cdot}25,$$

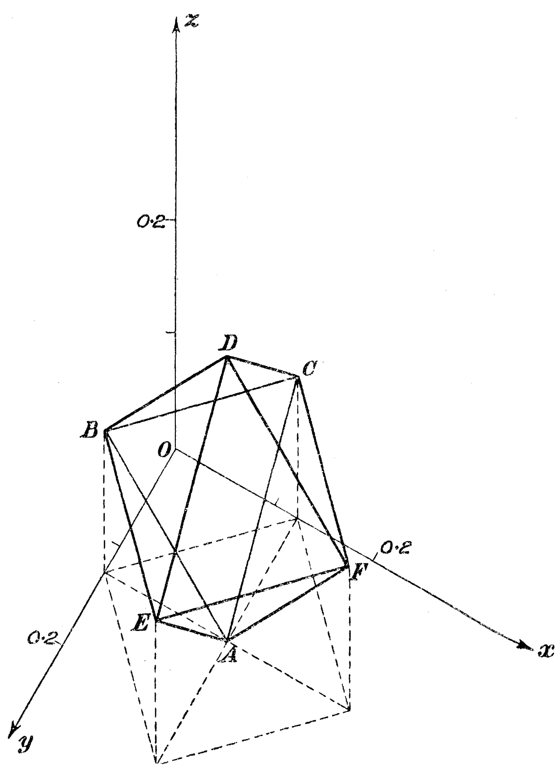


Fig. 10.

but do not come into account, the planes they represent only passing through the angles ABCDEF (fig. 10)* of the equilateral octahedron bounded by the planes—

* Figs. 10, 11, 12 are drawn to twice the scale of figures 1–5.

$$x + y + z = 0{\cdot}25 \qquad (\alpha)$$
$$x + y - z = 0{\cdot}25 \qquad (\beta)$$
$$x - y + z = 0{\cdot}25 \qquad (\gamma)$$
$$-x + y + z = 0{\cdot}25 \qquad (\delta)$$
$$x + y + z = 0{\cdot}5 \qquad (\epsilon)$$
$$x + y - z = 0 \qquad (\zeta)$$
$$x + y + z = 0 \qquad (\eta)$$
$$-x + y + z = 0 \qquad (\theta).$$

There are twelve somewhat interesting infinite series of definite inferences corresponding to the twelve sides of the octahedron. They may be grouped in three divisions following the contours EBCF, EDCA, BDFA, viz.:—

Given.	Inferred.
$x - y = 0{\cdot}125$ or $y - x = 0{\cdot}125$ or $x + y = 0{\cdot}125$ or $x + y = 0{\cdot}375$	$z = 0{\cdot}125$

for the contour EBCF, and the two similar systems that may be written down by cyclical substitution. These are all, it will be seen, inferences of *independence* ($z = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$), and therefore somewhat strikingly different to the usual definite inferences which are all inferences of complete association (all A's are B — no A's are B). From the same values of x and y we could, of course, infer $w = {\cdot}125$, so the theorem may be expressed in words thus—

"In any case where cross-equality* and independence both subsist for the second-order frequencies, independence must subsist for two at least of the four positive third-order frequencies if either (1) the sum of the remaining two is equal to $\frac{1}{8}$th or $\frac{3}{8}$ths of the total frequency; or (2) the difference between the remaining two is equal to $\frac{1}{8}$th of the total frequency."

I should, perhaps, note that in the present case the fact that the criterion of independence holds for the positive fourth-order class necessitates its holding for all the remaining classes of the aggregate. This is not so in general. It may further be remarked that the independence of the attributes, pair and pair in the second-order classes, does not connote independence for the groups of the third order. I hope to return to the logic of independence on a future occasion.

§ 29. Case (2)—

$$(AB)/(U) = (BC)/(U) = \&c. = q = 0{\cdot}20.$$

The pairs of attributes are now all *negatively* associated.

* I propose to use this term in lieu of the more lengthy "equality of contraries

The planes representing conditions of inferior congruence are

$$x = 0 \quad y = 0 \quad z = 0$$
$$x = 0{\cdot}1 \quad y = 0{\cdot}1 \quad z = 0{\cdot}1.$$

The conditions of superior congruence are represented by planes

$$x + y + z = 0{\cdot}1 \qquad (\alpha)$$
$$x + y - z = x - y + z = -x + y + z = 0{\cdot}2 \qquad (\beta, \gamma, \delta)$$
$$x + y + z = 0{\cdot}2 \qquad (\epsilon)$$
$$x + y - z = x - y + z = -x + y + z = -0{\cdot}1 \qquad (\zeta, \eta, \theta),$$

but on drawing the figure (fig. 11) it will be seen that the planes $(\beta, \gamma, \delta, \zeta, \eta, \theta)$ do not come into account, the surface being an octahedron bounded by (α) (ϵ) and the six planes representing conditions of inferior congruence.

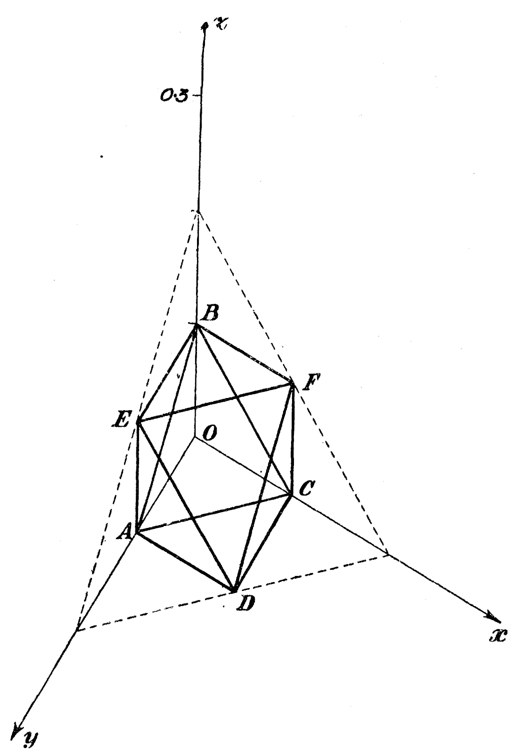


Fig. 11.

There are no infinite series of definite inferences but only six special cases, corresponding to the points ABCDEF. They are all of the form

	Given.		Inferred.
	$x = 0$	$y = 0$	$z = 0{\cdot}1.$
or	$x = 0{\cdot}1$	$y = 0{\cdot}1$	

It must be noted that in the type of cases at present under discussion q cannot be less than $\frac{1}{6}$th, or 0·166 . . . , for, by the *third*-degree conditions of consistence,

$$(AB) + (AC) + (BC) \nless (A) + (B) + (C) - (U)$$

that is if

$$(AB) = (AC) = (BC) = q \,.\, (U)$$
$$(A) = (B) = (C) = \tfrac{1}{2} \,.\, (U)$$
$$3q \nless \tfrac{1}{2}, \qquad\qquad q \nless \tfrac{1}{6}.$$

For this value of q the planes (α) and (ϵ) fall together into the origin; the whole surface, so to speak, closes up.

§ 30. Case (3)—

$$(AB)/(U) = (BC)/(U) = \&c. = 0·3.$$

In this case the pairs of attributes are all *positively* associated. The first case is therefore intermediate between the second case and the present one.

The planes representing the conditions of inferior congruence are

$$x = 0·1 \quad y = 0·1 \quad z = 0·1$$
$$x = 0·3 \quad y = 0·3 \quad z = 0·3,$$

but, as in Case (1), these do not actually come into account, only passing through the *edges* AC, BP, GD, and FM of the octahedron bounded by planes

$$x + y + z = 0·4 \qquad (\alpha)$$
$$x + y - z = x - y + z = -x + y + z = 0·3 \quad (\beta, \gamma, \delta)$$
$$x + y = z = 0·8 \qquad (\epsilon)$$
$$x + y - z = x - y + z = -x + y + z = 0·1 \quad (\zeta, \eta, \theta).$$

The surface is represented in fig. 12 below. To facilitate identification, but avoid overcrowding the small-scale figure with lettering, the planes may be stated to be

α . . .	ABC	ϵ . . .	EDF.
β . . .	GDFMH	ρ . . .	PQRCB.
γ . . .	QRMFE	η . . .	PBAHG.
δ . . .	PQEDG	θ . . .	CRMHA.

There are eighteen infinite series of definite inferences corresponding to the eighteen edges, all reducible to one or other of the types

Given.	Inferred.
$\pm (x - y) = 0·1$	$z = 0·2$
$x = 0·3$	$z = y$
$x + y = 0·25$	$z = 0·15$
$x + y = 0·55$	$z = 0·25.$

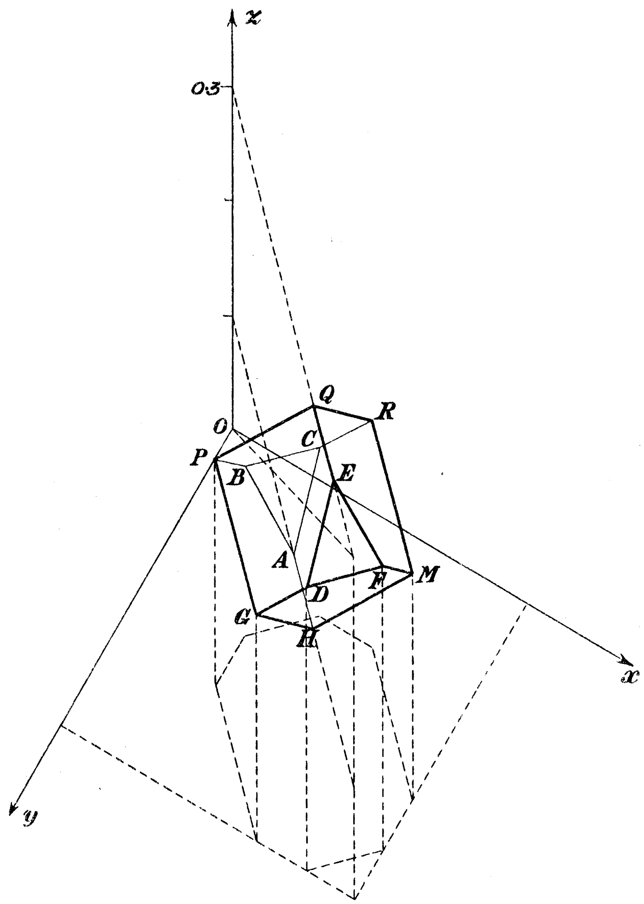


Fig. 12.

§ 31. For the benefit of those who would like to construct actual models of the surfaces shown in figs. 10, 11, 12, I again give below sketches of their developments, figs. 13—15. These sketches are drawn to half the scale of the preceding projections; the notation for the angles is the same as in figs. 6–9.

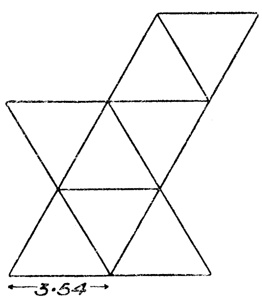


Fig. 13.

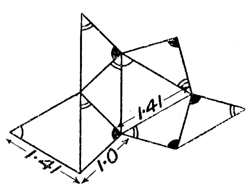


Fig. 14.

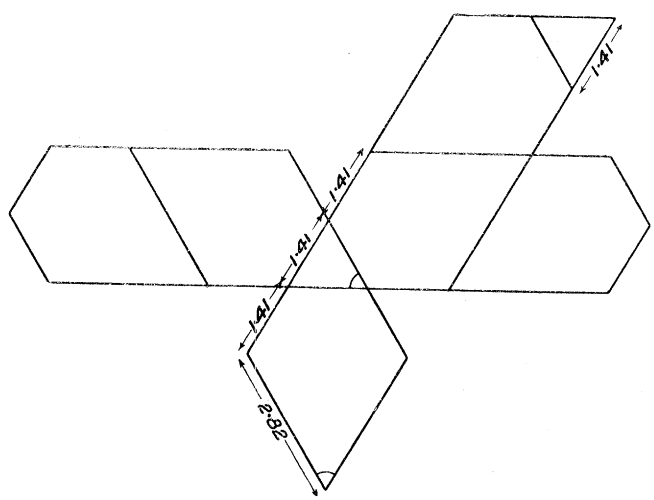


Fig. 15.

The Limits to Associations given by Conditions of Consistence.

§ 32—§ 37.

§ 32. The general conditions of consistence give limits to the frequencies of any one aggregate in terms of the frequencies of two or more given aggregates of the same congruence. Hence they give limits also to the possible associations between attributes in the unknown aggregate.

Thus, to take an imaginary example, suppose we have

(AB)	11	(AC)	8
$(A\beta)$	22	$(A\gamma)$	25
(αB)	39	(αC)	52
$(\alpha\beta)$	28	$(\alpha\gamma)$	15
	100		100.

Required the limiting values of the BC aggregate. From the conditions of consistence, § 5, we have

$$
\begin{aligned}
(BC) &\nless 33 + 50 + 60 - 100 - 11 - 8 \\
&\nless 24 \\
&\nless 11 + 8 - 33 \nless -14 \\
&\ngtr 60 + 11 - 8 \ngtr 63 \\
&\ngtr 50 - 11 + 8 \ngtr 47.
\end{aligned}
$$

Therefore the limiting values to (BC) are 24 and 47. But we know (B) = 50, (C) = 60, (U) = 100, therefore the limiting values to the frequencies of the aggregate are

(BC)	24	47
$(\beta\gamma)$	26	3
(βC)	36	13
$(\beta\gamma)$	14	37
	100	100.

Hence we may calculate, if desired, the limiting values to the association $|BC|$. For the association coefficient suggested in my previous memoir the values are $-0{\cdot}47$ and $-0{\cdot}96$. The values of $|AB|$ and $|AC|$ are $-0{\cdot}47$ and $-0{\cdot}83$ respectively.

For the great majority of cases occurring in practice the limits thus inferred from known associations are, as in the imaginary example, pretty wide. Very high values must be assigned to the two given associations before it is possible to infer even the sign of the third. Arguments of the vague type, "so many A's are B, and so many A's are also C, that clearly we must expect to find B and C frequently occurring together," are not uncommon, but the speaker seldom has any conception of the limits to (BC) actually implied by given values of (AB) and (AC).

§ 33. To enforce the danger of rashly inferring, we take some figures from the illustrations of my previous memoir borrowed from the material of the Childhood Society.

The following are the proportions,* per 10,000 cases observed, of those with given defects and given combinations of defects, for boys and girls of all ages.

A = development defects, B = nerve signs, C = low nutrition, D = mental dulness.

	Boys.	Girls.
(U)	10,000	10,000
(A)	878	682
(B)	1,085	850
(C)	285	325
(D)	789	689
(AB)	296	248
(AC)	142	180
(AD)	296	307
(BC)	134	141
(BD)	455	363
(CD)	123	132
(ABC)	57	66
(ABD)	153	128
(ACD)	179	80
(BCD)	64	63
(ABCD)	30	33

These figures give the following values for the associations†—

	Boys.	Girls.
|AB| . . .	0·750	0·784
|AC| . . .	0·848	0·916
|AD| . . .	0·846	0·900
|BC| . . .	0·783	0·814
|BD| . . .	0·897	0·905
|CD| . . .	0·823	0·835

A hasty arguer might think he was safe in inferring from the values, *e.g.*, of |BC| and |AC| that at least some A's must be B, if not that A and B were

* "On the Association of Attributes in Statistics," &c., 'Phil. Trans.,' A, vol. 194. Table on p. 318, but the figures reduced to proportion per 10,000 cases observed.

† Pages 306–307 of the same memoir.

positively associated. Either inference would be quite incorrect. Let us proceed to discuss the conditions of consistence.

§ 34. We have shown (§ 8) that a congruence of the fourth degree and second order is self-consistent when the conditions of consistence hold for each of the four congruences of the third degree into which it may be resolved. These conditions are as follows in the present instance; writing for brevity

$$x_1 = (AB)/(u) \qquad x_2 = (AC)/(u) \qquad x_3 = (AD)/(u)$$
$$x_4 = (BC)/(u) \qquad x_5 = (BD)/(u) \qquad x_6 = (CD)/(u).$$

Congruence. I. Boys.

ABC
$$x_1 = 0 \qquad x_2 = 0 \qquad x_4 = 0$$
$$x_1 = 0{\cdot}0878 \qquad x_2 = 0{\cdot}0285 \qquad x_4 = 0{\cdot}0285$$
$$(x_1 + x_2 + x_4 = -0{\cdot}7752)$$
$$x_1 + x_2 - x_4 = 0{\cdot}0878$$
$$x_1 - x_2 + x_4 = 0{\cdot}1085$$
$$-x_1 + x_2 + x_4 = 0{\cdot}0285$$

ABD
$$x_1 = 0 \qquad x_3 = 0 \qquad x_5 = 0$$
$$x_1 = 0{\cdot}0878 \qquad x_3 = 0{\cdot}0789 \qquad x_5 = 0{\cdot}0789$$
$$(x_1 + x_3 + x_5 = -0{\cdot}7248)$$
$$x_1 + x_3 - x_5 = 0{\cdot}0878$$
$$x_1 - x_3 + x_5 = 0{\cdot}1085$$
$$-x_1 + x_3 + x_5 = 0{\cdot}0789$$

ACD
$$x_2 = 0 \qquad x_3 = 0 \qquad x_6 = 0$$
$$x_2 = 0{\cdot}0285 \qquad x_3 = 0{\cdot}0789 \qquad x_6 = 0{\cdot}0285$$
$$(x_2 + x_3 + x_6 = -0{\cdot}8051)$$
$$x_2 + x_3 - x_6 = 0{\cdot}0875$$
$$x_2 - x_3 + x_6 = 0{\cdot}0285$$
$$-x_2 + x_3 + x_6 = 0{\cdot}0789$$

BCD
$$x_4 = 0 \qquad x_5 = 0 \qquad x_6 = 0$$
$$x_4 = 0{\cdot}0285 \qquad x_5 = 0{\cdot}0789 \qquad x_6 = 0{\cdot}0285$$
$$(x_4 + x_5 + x_6 = -0{\cdot}7841)$$
$$(x_4 + x_5 - x_6 = 0{\cdot}1085)$$
$$x_4 - x_5 + x_6 = 0{\cdot}0285$$
$$-x_4 + x_5 + x_6 = 0{\cdot}0789.$$

Congruence. II. Girls.

ABC
$$x_1 = 0 \qquad x_2 = 0 \qquad x_4 = 0$$
$$x_1 = 0{\cdot}0682 \qquad x_2 = 0{\cdot}0325 \qquad x_4 = 0{\cdot}0325$$
$$(x_1 + x_2 + x_4 = -0{\cdot}8142)$$
$$x_1 + x_2 - x_4 = 0{\cdot}0682$$
$$x_1 - x_2 + x_4 = 0{\cdot}0850$$
$$-x_1 + x_2 + x_4 = 0{\cdot}0325$$

Congruence. II. Girls.

ABD

$$x_1 = 0 \qquad x_3 = 0 \qquad x_5 = 0$$
$$x_1 = 0{\cdot}0682 \qquad x_3 = 0{\cdot}0682 \qquad x_5 = 0{\cdot}0689$$
$$(x_1 + x_3 + x_5 = -\ 0{\cdot}7779)$$
$$x_1 + x_3 + x_5 = 0{\cdot}0682$$
$$x_1 - x_3 + x_5 = 0{\cdot}0850$$
$$-\ x_1 + x_3 + x_5 = 0{\cdot}0689$$

ACD

$$x_2 = 0 \qquad x_3 = 0 \qquad x_6 = 0$$
$$x_2 = 0{\cdot}0325 \qquad x_3 = 0{\cdot}0682 \qquad x_6 = 0{\cdot}0325$$
$$(x_2 + x_3 + x_6 = -\ 0{\cdot}8304)$$
$$x_2 + x_3 - x_6 = 0{\cdot}0682$$
$$x_2 - x_3 + x_6 = 0{\cdot}0325$$
$$-\ x_2 + x_3 + x_6 = 0{\cdot}0689$$

BCD

$$x_4 = 0 \qquad x_5 = 0 \qquad x_6 = 0$$
$$x_4 = 0{\cdot}0325 \qquad x_5 = 0{\cdot}0689 \qquad x_6 = 0{\cdot}0325$$
$$(x_4 + x_5 + x_6 = -\ 0{\cdot}8136)$$
$$x_4 + x_5 - x_6 = 0{\cdot}0850$$
$$x_4 - x_5 + x_6 = 0{\cdot}0325$$
$$-\ x_4 + x_5 + x_6 = 0{\cdot}0689.$$

§ 35. If the limits to each class frequency given by these relations be worked out it will be found that the lower limit to every class, in terms of the others, is *zero* without exception. That is to say, any pair whatever of the given defects might exhibit complete "disassociation" (association coefficient $= -1$), without this being in any way inconsistent with the high associations exhibited by other pairs. In two cases for the boys and three for the girls *upper* limits could, however, be inferred, as given below.

Group.	Limit given by congruence.	Limits.		Corresponding associations.	
		Boys.	Girls.	Boys.	Girls.
(AB) . .	ABC	870	643	0·9996	0·9972
(AD) . .	ABD	630	574	0·9861	0·9953
,, . .	ACD	—	634	—	0·9991
(BD) . .	ABD	—	630	—	0·9955

Thus our imaginary "hasty arguer," if he attempted to infer from the given values of (BC) and (AC) that "some A's must be B" or, worse still, that "A and B must be

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

positively associated," would be inferring almost the direct contrary of the truth. The real inference is "some A's are *not* B," or "A and B are not completely associated." Similarly from the given values of (AB) and (BD), or of (AC) and (CD) in the case of the girls, he could only infer "some D's are not A"; and from (AB) and (BD) in the case of the girls again, "some D's are not B."

§ 36. In order to illustrate the case completely I give sketches of the congruence-surfaces in figures 16—19 for the boys and 20—23 for the girls,* and have marked in each figure the co-ordinates corresponding to the *actual* values of (AB)/(*u*), &c.

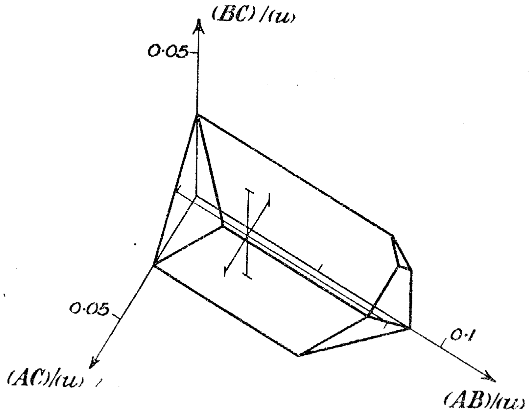


Fig. 16.

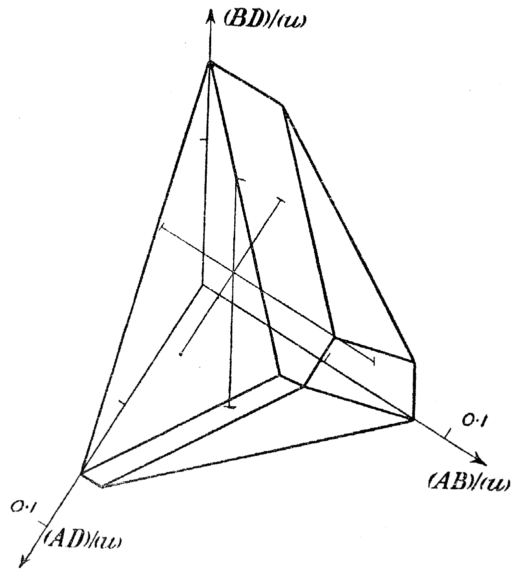


Fig. 17.

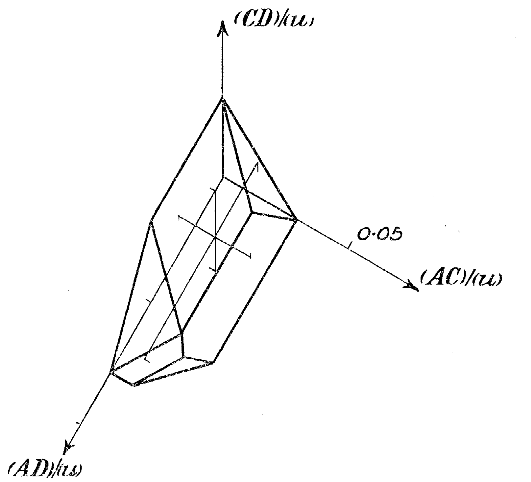


Fig. 18.

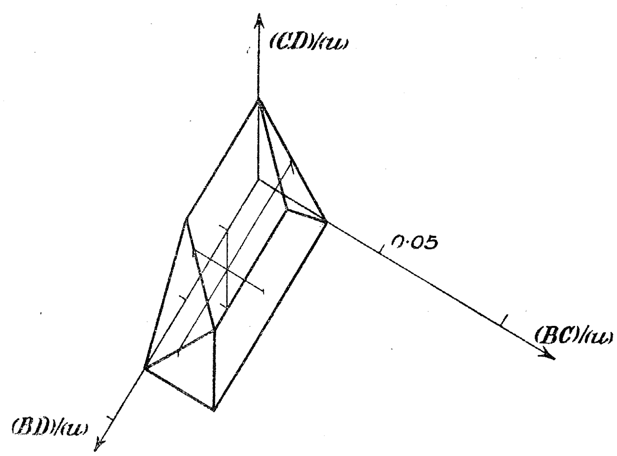


Fig. 19.

* Note that the figures are on a much larger scale than those previously given. In figs. 21 and 22 the width of the narrow strip cut off on the plane (AD)/(*u*) = ·0682 has been somewhat exaggerated.

Only in the case of figs. 17 and 21 does the general appearance of the surface resemble that shown in fig. 4; in all the other cases the greater part of the surface is given by the conditions of inferior congruence. Mere inspection of the figures shows

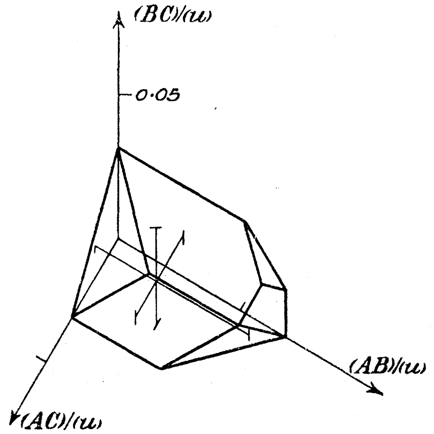


Fig. 20.

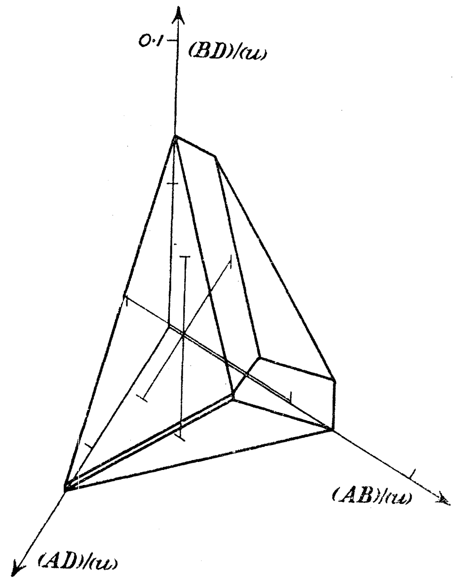


Fig. 21.

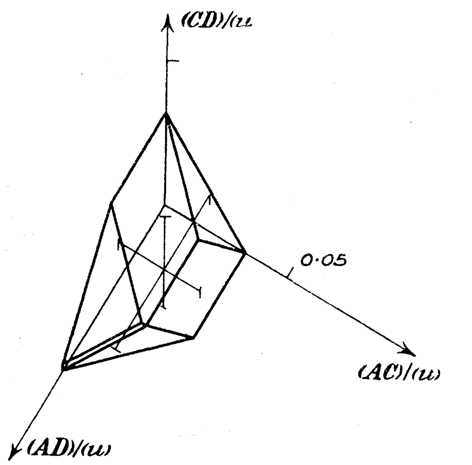


Fig. 22.

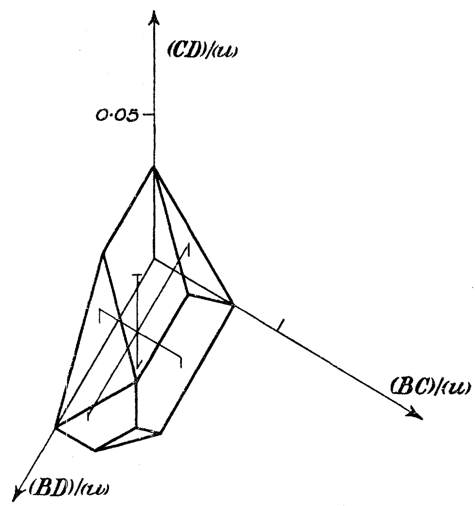


Fig. 23.

that while, *e.g.*, a very slight alteration of the actual values of (BC) and (AC) would lead to a lower limit for (AB) (*cf.* figs. 16 and 20) the associations between A and B and A and C would have to be very largely increased indeed before we could infer (BC) $>$ 0. A study of such figures as these would, I think, lead to a good deal more caution in statistical inference.

§ 37. It may be remarked that while the conditions of inferior congruence for second-order groups always give the limits ± 1 to the corresponding associations,

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

the inferior conditions for the groups of third or higher orders only assign these extreme values to some, but not all, of the partial coefficients. Thus, for example, take the figures of § 32. Applying the inferior conditions of consistence (§ 5, II.) we find as limits to (ACD) for the boys 0 and 123. The remaining frequencies of the aggregate must then be—

(ACD)	0	123
(ACδ)	142	19
(AγD)	296	173
(αCD)	123	0
(αγD)	370	493
(αCδ)	20	143
(Aγδ)	440	563
(αγδ)	8609	8486.

The aggregate corresponding to the minimum value of (ACD) will evidently give minimum values to the partial associations in *positive* universes like $|AC|D|$, but maximum values to those in negative universes $|AC|\delta|$, $AD|\gamma|$, &c. Then inspection will show that only for $|AC|D|$ and $|AD|C|$ are the limits ± 1; for the remaining associations the limits are—

$\|CD\|A\|$	-1	$+\cdot 91$
$\|AC\|\delta\|$	$+\cdot 33$	$+\cdot 99$
$\|AD\|\gamma\|$	$+\cdot 68$	$+\cdot 88$
$\|CD\|\alpha\|$	-1	$+\cdot 99.$

In the discussion of the Childhood Society's material in my previous memoir I remarked on the fact that all the partial coefficients of association between defects in positive universes were small, while those in negative universes were large and positive. I at first thought that this might be a *logical* consequence of the given values of the second-order groups, but this is not the case. The values of the associations like $|AC|D|$, $|AD|C|$, &c., are almost indeterminate; certain of the coefficients with negative universes ($|AC|\delta|$, $|AD|\gamma|$) are necessarily positive, but others ($|CD|\alpha|$) may fall to the extreme limit -1.

§ 38. It may be useful to remark that if two associations $|AB|$ and $|AC|$, in a third-order congruence, are both equal to zero, the limits to the third association $|BC|$ are necessarily ± 1, *whatever the values* of p_1, p_2, p_3. If we write $p_1 p_2$ for $(AB)/(U)$ or x, $p_1 p_3$ for $(AC)/(U)$ or y, the limits to $(BC)/(U)$ or z are

$$z \not< (1-p_1)(p_2+p_3-1) \quad (1)$$
$$\not< \quad p_1 (p_2+p_3-1) \quad (2)$$
$$\not> \quad p_2 + p_1 (p_3 - p_2) \quad (3)$$
$$\not> \quad p_3 + p_1 (p_2 - p_3) \quad (4).$$

Now if $p_2 + p_3 > 1$ the lower limit to z is, by the inferior conditions of consistence, $p_2 + p_3 - 1$. But (1) and (2) both give lower limits still, and therefore do not come into account. Again, if $p_2 + p_3 < 1$, the lower limit to z, by the inferior conditions, is zero. But (1) and (2) give negative limits, and therefore again do not come into account.

As regards the major limits, z must not, by the inferior conditions, be greater than the least of p_2 and p_3. But if p_2 be the less (3) and (4) both give limits greater than p_2; if p_3 be the less (3) and (4) both give limits greater than p_3. In neither case then do they come into account. We have therefore proved the theorem.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES